

# The Mathematics Enthusiast

---

Volume 7  
Number 2 *Numbers 2 & 3*

Article 21

---

7-2010

## TME Volume 7, Numbers 2 and 3

Follow this and additional works at: <https://scholarworks.umt.edu/tme>



Part of the [Mathematics Commons](#)

**Let us know how access to this document benefits you.**

---

### Recommended Citation

(2010) "TME Volume 7, Numbers 2 and 3," *The Mathematics Enthusiast*. Vol. 7 : No. 2 , Article 21.

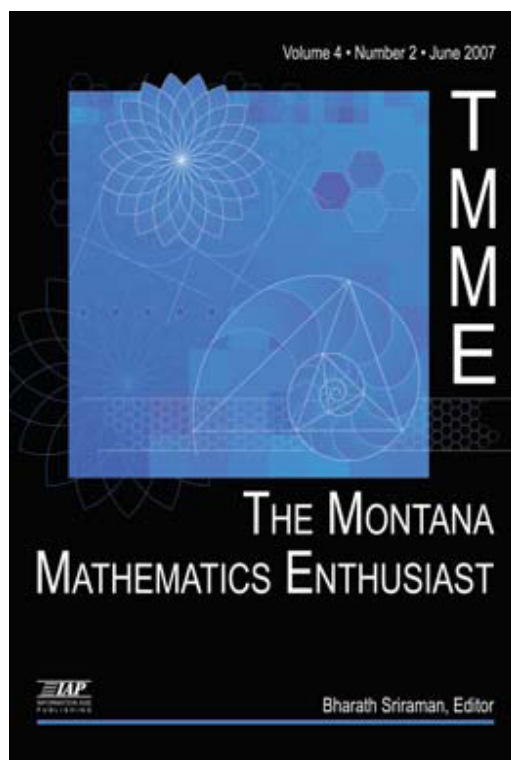
Available at: <https://scholarworks.umt.edu/tme/vol7/iss2/21>

This Full Volume is brought to you for free and open access by ScholarWorks at University of Montana. It has been accepted for inclusion in The Mathematics Enthusiast by an authorized editor of ScholarWorks at University of Montana. For more information, please contact [scholarworks@mso.umt.edu](mailto:scholarworks@mso.umt.edu).

# The Montana Mathematics Enthusiast

ISSN 1551-3440

VOL. 7, NOS 2&3, JULY 2010, pp.175-462



**Editor-in-Chief**

**Bharath Sriraman, The University of Montana**

**Associate Editors:**

**Lyn D. English, *Queensland University of Technology, Australia***

**Simon Goodchild, *University of Agder, Norway***

**Brian Greer, *Portland State University, USA***

**Luis Moreno-Armella, *Cinvestav-IPN, México***

### **International Editorial Advisory Board**

- Mehdi Alaeiyan, *Iran University of Science and Technology, Iran*  
Miriam Amit, *Ben-Gurion University of the Negev, Israel*  
Ziya Argun, *Gazi University, Turkey*  
Ahmet Arikan, *Gazi University, Turkey.*  
Astrid Beckmann, *University of Education, Schwäbisch Gmünd, Germany*  
Raymond Bjuland, *University of Stavanger, Norway*  
Morten Blomhøj, *Roskilde University, Denmark*  
Robert Carson, *Montana State University- Bozeman, USA*  
Mohan Chinnappan, *University of Wollongong, Australia*  
Constantinos Christou, *University of Cyprus, Cyprus*  
Bettina Dahl Søndergaard, *University of Aarhus, Denmark*  
Helen Doerr, *Syracuse University, USA*  
Ted Eisenberg, *Ben-Gurion University of the Negev, Israel*  
Paul Ernest, *University of Exeter, UK*  
Viktor Freiman, *Université de Moncton, Canada*  
Fulvia Furinghetti, *Università di Genova, Italy*  
Anne Birgitte Fyhn, *Universitetet i Tromsø, Norway*  
Eric Gutstein, *University of Illinois-Chicago, USA*  
Marja van den Heuvel-Panhuizen, *University of Utrecht, The Netherlands*  
Gabriele Kaiser, *University of Hamburg, Germany*  
Libby Knott, *Washington State University, USA*  
Yasemin Kıymaz, *Ahi Evran Üniversitesi, Turkey*  
Tinne Hoff Kjeldsen, *Roskilde University, Denmark*  
Jean-Baptiste Lagrange, *IUFM-Reims, France*  
Stephen Lerman, *London South Bank University, UK*  
Frank Lester, *Indiana University, USA*  
Richard Lesh, *Indiana University, USA*  
Claus Michelsen, *University of Southern Denmark, Odense*  
Reidar Mosvold, *University of Stavanger*  
Nicholas Mousoulides, *University of Cyprus, Cyprus*  
Swapna Mukhopadhyay, *Portland State University, USA*  
Norma Presmeg, *Illinois State University, USA*  
Gudbjorg Palsdottir, *University of Iceland*  
João Pedro da Ponte, *University of Lisbon, Portugal*  
Demetra Pitta Pantazi, *University of Cyprus, Cyprus*  
Linda Sheffield, *Northern Kentucky University, USA*  
Olof Bjorg Steinhorsdottir, *University of North Carolina- Chapel Hill, USA*  
Günter Törner, *University of Duisburg-Essen, Germany*  
Renuka Vithal, *University of KwaZulu-Natal, South Africa*  
Dirk Wessels, *UNISA, South Africa*  
Nurit Zehavi, *The Weizmann Institute of Science, Rehovot, Israel*

*The Montana Mathematics Enthusiast* is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy. The journal is published by Information Age Publishing and the electronic version is hosted jointly by IAP and the Department of Mathematical Sciences- The University of Montana, on behalf of MCTM. Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning.

The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at <http://www.montanamath.org/TMME> or <http://www.math.umt.edu/TMME/>

#### **Indexing Information**

*Australian Education Index;*

*EBSCO Products (Academic Search Complete);*

*EDNA;*

*Cabell's Directory of Publishing Opportunities in Educational Curriculum and Methods*

*Directory of Open Access Journals (DOAJ);*

*Inter-university Centre for Educational Research (ICO)*

*PsycINFO (the APA Index);*

*MathDI/MathEDUC (FiZ Karlsruhe);*

*Journals in Higher Education (JIHE);*

*Ulrich's Periodicals Directory;*

*Zentralblatt MATH*



# THE MONTANA MATHEMATICS ENTHUSIAST

## ISSN 1551-3440

Vol.7, Nos.2&3, July 2010, pp.175-462

*“We publish while others perish”*

- Howard Zinn (1922-2010)

### TABLE OF CONTENTS

#### 0. EDITORIAL

So many journals,  
So many words...  
So what?

Bharath Sriraman (USA).....pp.175-176

#### FEATURE ARTICLES

1. Women Belonging in the Social Worlds of Graduate Mathematics  
Abbe H. Herzig (USA).....pp.177-208
2. Magic Circles in the Arbelos  
Christer Bergsten (Sweden).....pp.209-222
3. Chaos in Physics and recurrence in Arithmetic Sets.  
Jean Dayantis (France).....pp.223-246
4. Polysemy of symbols: Signs of ambiguity  
Ami Mamolo (Canada).....pp.247-262
5. Cultural conflicts in mathematics classrooms and resolution: The case of immigrants from the  
Former Soviet Union and Israeli “Old timers”  
Miriam Amit (Israel).....pp.263-274

6. Climbing and Angles: A Study of how two Teachers Internalize and Implement the Intentions of a Teaching Experiment  
Anne Birgitte Fyhn (Norway).....pp.275-294
7. Vectors in climbing  
Anne Birgitte Fyhn (Norway).....pp.295-306
8. Rules Without Reason: Allowing Students to Rethink Previous Conceptions  
Mary Mueller, Dina Yankelewitz, Carolyn Maher (USA).....pp.307-320
9. A semiotic reflection on the didactics of the Chain rule  
Omar Hernández Rodríguez and Jorge M. López Fernández (Puerto Rico).....pp.321-332
10. Some reflections on Hernández and López's reflections on the chain rule  
Michael N. Fried (Israel).....pp.333-338

### **Montana Feature**

11. The history of mathematics as a pedagogical tool: Teaching the integral of the secant via Mercator's projection  
Nick Haverhals & Matt Roscoe (Montana).....pp.339-368

### **Regional Feature**

#### **Northwest (U.S) Women Researchers in Math and Science Education**

12. Guest Editorial: A Starting Point  
Ke Wu Norman (Montana).....pp.369-370
13. Rehearsal or Reorganization: Two Patterns of Literacy Strategy Use in Secondary Mathematics Classes  
Anne Adams (Idaho).....pp.371-390
14. Pre-service Teachers in Mathematics Lesson Study  
Elizabeth A. Burroughs & Jennifer L. Luebeck (Montana).....pp.391-400
15. Internal and External Comments on Course Evaluations and Their Relationship to Course Grades  
Hilary Smith Risser (Montana).....pp.401-412

16. Problem Posing from the Foundations of Mathematics Libby Knott (Washington).....	pp.413-432
17. A Graduate Level In-Service Teacher Education Curriculum Integrating Engineering into Science and Mathematics Contents Ke Wu Norman (Montana) Tamara J. Moore (Minnesota) & Anne L. Kern (Idaho).....	pp.433-446
18. The Influence of a Multidisciplinary Scientific Research Experience on Teachers Views of Nature of Science Jerine Pegg (Alberta) & Edith Gummer (Oregon).....	pp.447-460
19. EPILOGUE Bharath Sriraman (Montana).....	pp.461-462





**So many journals,  
So many words...  
So what?**

*Bharath Sriraman*

As I was putting the finishing touches to this journal issue, my 6-year old posed the following question, who reads all this stuff, Dad? It was a simple question that merited an answer that was not pedantic but understandable to someone viewing the whole enterprise of publishing with innocent eyes. This in turn led me to ponder over publishing which characterizes academia and the culture of many academic institutions. The ensuing dialogue proceeded as follows:

Do you like doing math in school?

*Yes, it is fun and also easy. I like big numbers...Once in the school bus I counted up to 400.*

Why 400?

*I had to get off. We can also go up to infinity, or infinity plus one and infinity plus two*

That's good. All these pages are about math and also about making it fun and easy for those that think it is hard

*It's a lot of pages*

Sometimes I wonder about how many pages have been written in all and if anyone reads anything

*How many?*

That is a really good question. I will think of a way to make an estimate

*A million?*

I don't know...it is frightening to think about this!

The question is wide open for the field. How many pages per year of mathematics education related writing is published each year? More precisely, how many mathematics education "research results" are published each year? This is somewhat analogous to the question and estimate of Stanislaw Ulam about the number of results/theorems in mathematics that were produced each year [His estimate was approximately 100,000 back in the 1970's]. More importantly, what proportion of this mathematical drivel (regardless of whether it is pure or applied) is meaningful, mutates and survives, as opposed to perishing in esoteric and dusty journal collections in libraries? To answer this question, as Davis & Hersh (1981) pointed out, one would have to set up a filter to sieve out what is important as opposed to technical gobbledygook, and the experts in the field would have to impose a value judgment on what is a significant result as opposed to yet another clever proof to an old problem or another glorified lemma. Such a Darwinian view might be seem alarming to the reader and trivialize what we do, but the fact of the matter is that we as a field have seen a geometric increase in the number of outlets for mathematics education research and practice. We have research journals, practitioner journals, hybrid journals (such as *The Montana Mathematics Enthusiast*) that engage in ideas in mathematics as well as mathematics education, and expository journals such as *The Mathematical Intelligencer*, *The Mathematical Gazette* and others that make mathematics accessible to the lay audience. All this is in addition to conference proceedings, symposium proceedings, edited books in different book series, Handbooks of all sorts, and last but not least the steady output in masters theses and doctoral dissertations! I have been kind enough not to include grant proposals, textbooks and other teacher related material that is produced anew each year.

A related question in the eco-conscious times that we live would be: What is the carbon footprint of mathematics education as a field? Is the whole enterprise of supporting the lumber and paper pulp industry, as well as publishing houses resulting in *anything* at all? What do I mean by "anything?" Those looking into the field of mathematics education typically associate "anything" with change of some sort- either curricular change, change in achievement scores, change in teachers, change in students, change in the number of students entering STEM fields, change in minority achievement,

change in status of a country on international comparison tests, and “change” typically implies something measurable or something that can be quantified. Indeed mathematics (education) has even become a status symbol in some countries. In December 2001, after the results of PISA 2000 became public, a left leaning periodical in Germany called *Der Spiegel* had a front cover that stated: **Sind deutsche Schüler doof?**<sup>1</sup>, which translates to “Are German students stupid?”. This rhetorical question was based on the poor performance of German students on the PISA test and the erroneous notion of “national pride” being lost on account of a test! There are similar (not necessarily more sensitively worded) headlines in other parts of the world related to other international comparisons tests such as *Finland finishes first, Singapore students are the best, Icelandic girls outperform boys....*and so on. Politicians and policy makers are very quick to jump on the voter cache of such alarming proclamations, and national funding bodies follow *en suite* (pun intended) by providing grants for “changing” curriculum, comparative studies, new “innovative” teacher education programs, professional development etc. This in turn leads many researchers in our field to jump on the funding wagon for the sake of institutional prestige, professional monikers, and engage in all kinds of projects, which I won’t bother to list, and this neatly feeds into the publishing outlets I alluded to earlier. The cycle of student measurement on tests, blaming schools and teachers, and research related to all this continues unabated contributing to the well being of the publishing industry of which we are all a part of. A lot of this criticism is self directed since the journal is a tiny cog in the giant academic-publishing industry. Perhaps we need to re-examine or re-state the questions we are asking in the first place?

Is it really the burden of our field (here I mean mathematics education) to initiate “anything”, i.e., so many changes, many of which are of significant social magnitude? I do not see geography or biology or art burdened with such expectations. To this many will give their glib arguments about the centrality and significance of mathematics, and how it is essential/applicable to other fields, in order to lend credence to their own enterprises. I do not agree with such utilitarian claims. The discipline is beautiful and significant in its own right but it assumes a life of its own when Society, Institutions and policy makers start ascribing pure objectivity to it as well as use it as a sieve to justify decision making and stratification of society through high stakes examinations. If there is any change that occurs at all, it ought to occur locally, within the minds of a student exposed to the subject, to perhaps experience insight, meaning? If this is the “basic” purpose as some put it, I suggest looking up the number of pages produced each year on answering the search for meaning...we as a field pale in comparison to output of the “search for meaning” industry. Our carbon footprint is insignificant to the self-help industry!

The journal publishes articles that are considered stimulating from a mathematical viewpoint, i.e., they are expository enough for teachers and undergraduate/beginning graduate university students to read and understand. They also tend to be eclectic, i.e., presenting mathematics in new contexts. Some articles deal with the historical development of mathematical ideas and some pieces tackle sociological issues that are related to teaching and learning, hopefully provocative from an educational viewpoint. Finally meaning cannot be derived nor imparted without an understanding of factors that impede or facilitate learning, which relates to the psychology of mathematics education. At the end of the day, the journal hopes that the articles stimulate readers to become more enthusiastic about mathematics, realize its presence in different contexts as well as become aware of the sociological issues surrounding its teaching and learning. If the journal results in enacting the reader’s sense of agency and promoting enthusiasm, activism or even an aesthetic, then we have achieved more than we set out to do. I am particularly pleased with the collection of articles that constitute vol7, nos2&3.

## References

- Davis, P., & Hersh, R., (1981). *The Mathematical Experience*. Birkhäuser, Boston.  
 Winfried, S. (2007). *Wer ist Alexander Grothendieck? Anarchie, Mathematik, Spiritualität – eine Biographie*. Havixbeck.

---

<sup>1</sup> <http://www.spiegel.de/schulspiegel/0,1518,172357,00.html>

## Women Belonging in the Social Worlds of Graduate Mathematics

Abbe H. Herzig<sup>1</sup>

University at Albany, State University of New York

*Abstract: The participation of women in post-graduate mathematics still lags substantially behind that of men. Drawing upon sociocultural theories of learning, I argue that success in graduate school necessitates learning mathematical content, participating in mathematical practices, and developing a sense of belonging in mathematics. Using an institutional ethnography approach, I interviewed 12 women graduate students from three mathematics departments in the U.S. to document their experiences within the social relations of graduate mathematics. They described both intrinsic and extrinsic obstacles to belonging, including a tension between their desire to belong and their needs to distance themselves from what they perceived to be the mathematical culture. These women's stories are interpreted in terms of the ways they are multiply "marked" as deviant (Damarin, 2000)—as women, as mathematically talented, and as women in mathematics; for women of color or mothers, these markings are even more complex.*

**Keywords:** belonging, graduate students, institutional ethnography, women

### 1. Introduction

Despite increasing attention over recent decades, women's participation in advanced mathematics in the U.S. still lags substantially behind that of men. In the U.S. in 2006, women earned 41% of bachelors degrees and 43% of masters degrees, comprised 32% of first-year, full-time graduate students (30% of total full-time students) in doctorate-granting mathematics departments, earned 32% of PhDs (among U.S. citizens, women earned only 27% of PhDs), received 22% of new doctoral positions in PhD granting departments, and comprised 12% of full-time tenured or tenure track faculty (25% of non-tenure-track faculty) at doctoral granting

---

<sup>1</sup> 8 Winter Haven  
West Sand Lake, NY 12196  
[aherzig@albany.edu](mailto:aherzig@albany.edu)  
phone: (518) 727-4755  
fax: (518) 308-0285

institutions (National Science Foundation, 2008; Phipps, Maxwell, & Rose, 2007). Admittedly, these statistics give only an approximate picture of how women fare as they progress along the path from college through work in the academic world, as these statistics represent different cohorts of individuals at one fixed point in time, but the statistical picture is compelling nonetheless, implying that the persistence of women in mathematics is a problem throughout and after graduate school.

Of course, there is more to women's experiences in graduate mathematics than even the numbers show. "The question is not only one of retention in doctoral study but the more subtle one of whether women have a graduate experience that is of as high a quality as that of men" (Etzkowitz, Kemelgor, Neuschatz, & Uzzi, 1992, p. 158). In this paper, I use an institutional ethnography approach (Smith, 2005; Campbell & Gregor, 2004) to examine the nature of women's experiences in graduate mathematics. It is not the goal here to compare women's experiences with those of men; indeed, many of the obstacles and issues women face likely affect men as well. However, particularly in the context of recent public concern about the small numbers of women persisting in mathematics and the sciences, this analysis was undertaken to add depth to our insights about women's experiences in mathematics and the reasons for their relatively slow progress into advanced positions in the field.

### *1.1 Learning Graduate Mathematics*

Theories of situated learning posit that learning happens through participation in social practices, and that learning is intertwined with, and inseparable from that participation (Boaler, 2002; Lave & Wenger, 1991; Rogoff, 1994; Wenger, 1998). For mathematics graduate students, learning happens as they participate in the communities of practice found in their programs and departments. Etienne Wenger (1998) describes three dimensions that define a community of practice: a joint enterprise, a shared repertoire, and mutual engagement. The *joint enterprise* is comprised of the activities in which the members of the community engage together. Although the enterprise may be circumscribed by forces that are beyond participants' immediate control, they mutually construct and define the enterprise as they pursue it. In doctoral mathematics, the joint enterprise is learning to become mathematicians, as the students interact with each other and with faculty to appropriate and develop mathematical knowledge of all sorts. A *shared repertoire* is similar to what Tomas Gerholm (1990) calls "tacit knowledge," the often unspoken norms and practices by which the discipline operates. For doctoral study in mathematics, the

shared repertoire includes all the practices that are inherent in enrollment in graduate school, such as studying for courses and exams, finding a research topic, and working on a dissertation. Both the joint enterprise and shared repertoire are constructed and negotiated by participants as they *mutually engage* in the activities of their community. These three dimensions of the community of practice of graduate school entail students' appropriation of mathematical knowledge (entering and constructing the joint enterprise), practices (entering and constructing the shared repertoire), and a sense of belonging within the discipline (engaging in mutual ways with the other community members) (Boaler, 2002; Boaler, Wiliam, & Zevenbergen, 2000; Herzig, 2004a). Students who have limited access to any of the three dimensions of learning of mathematics—acquiring knowledge, practices, and a sense of belonging—will be inhibited in their opportunities to learn and engage with mathematics, and will be less likely to persist in mathematics. Each of these dimensions is affected by students' interactions with other members of the community—the students and faculty. Those relationships, in turn, are formed by and contribute to the departmental and program structure, policies, and culture.

Beyond the communities of practice of graduate school itself, graduate students are also working to engage with the practices of mathematicians. Mathematics doctoral programs in the U.S. are primarily structured around providing disciplinary training in the core areas of mathematical scholarship (Bass, 2003; National Research Council, 1992). Although there are some notable exceptions, the first several years of doctoral education in mathematics typically follow a “transmission” model of teaching (Rogoff, 1994), in which faculty lecture, students take notes and study extensively outside of class, with most interactions between the two taking place as faculty grade assignments and exams (National Research Council, 1992). Consequently, many graduate students have few opportunities to participate in many of the activities of professional mathematicians. To the extent that a graduate program sequesters prospective mathematicians from the genuine practices of mathematicians, it limits their opportunities to learn to work as mathematicians (Herzig, 2002; Lave & Wenger, 1991). Hyman Bass (2003) argues that these programs need to do a better job of preparing students for all aspects of work within the profession of mathematics, including serious professional development for teaching, uses of technology, exposition, developing and pursuing a research program, participation in the local and broader mathematical communities, and development of a “cultural awareness in students of the significance of their discipline in the larger worlds of science and society and of

the expectation that they will serve as emissaries of their discipline in the outside world” (p. 775). While there has long been an emphasis on the acquisition of knowledge, Bass’s argument represents a more recent emphasis on graduate students’ need to learn the practices of the profession. However, little attention has been paid to students’ development of a sense that they have a place within the mathematical community.

### *1.2 Belonging in Mathematics*

Building students’ sense of *belongingness* in mathematics has been proposed as a critical feature of an equitable K-12 education (Alleksaht-Snider & Hart, 2001; Ladson-Billings, 1997; National Council of Teachers of Mathematics, 2000; Tate, 1995). Martha Alleksaht-Snider and Laurie Hart (2001) argue that when schooling facilitates all students’ sense of belongingness and engagement with mathematics, then we are more likely to achieve the goal of “mathematics for all” so often cited as a goal in reform and policy documents, and define *belonging* as “the extent to which each student *senses* that she or he belongs as an important and active participant” in mathematics (p. 97). A similar construct has been proposed at the doctoral level, with several authors arguing that students’ involvement or integration into the communities of their departments is important for their persistence (Girves & Wemmerus, 1988; Golde, 1996; Herzig, 2002, 2004a; Lovitts, 2001; National Research Council, 1992; National Science Foundation, 1998; Tinto, 1993). In particular, Vincent Tinto (1993) proposes that doctoral student persistence is a function of both social and academic integration within the communities of the local department or program. This extends Alleksaht-Snider and Hart’s (2001) definition to define belonging for a graduate student as her sense that she is an important and active participant in both the academic *and social* communities of her department and program (Herzig, 2006). In interviews with 18 graduate students in one mathematics doctoral program in the U.S., students who had multiple avenues to develop a sense of belonging within mathematics (for example, through family members who were mathematicians, or involvement in mathematics since a young age) were found to be more likely to persist through the Ph.D. (Herzig, 2002). It seems, then, that developing an identity as a mathematician, a sense that “I belong here,” is one critical component in the persistence of doctoral students.

### *1.3 Obstacles to Belonging for Women*

Many students face obstacles in graduate mathematics, including harsh weed-out policies and competition (Herzig, 2006; Hollenshead, Younce, & Wenzel, 1994; Stage & Maple, 1996),

pedagogy that fails to communicate the passion or depth of mathematics (Burton, 1999; Herzig, 2002; Stage & Maple, 1996), and limited or negative relationships with advisors and other faculty (Bair & Haworth, 1999; Etzkowitz, Kemelgor, & Uzzi, 2000; Girves & Wemmerus, 1988; Golde, 1996; Herzig, 2004b).

Women and people of color face additional obstacles. Women in science have experienced discrimination in finding and working with mentors and been excluded from the informal social networks of their laboratories or departments, treated as “invisible,” or otherwise had their contributions marginalized (Becker, 1990; Committee on the Participation of Women, 2003; Etzkowitz et al., 1992; Etzkowitz et al., 2000; Sonnert & Holton, 1995; Stage & Maple, 1996). In mathematics in particular, women have reported blatantly sexist behavior, including unwanted sexual advances from faculty, tolerance of public sexist comments, and professors who openly state that women are not as smart, dedicated, or talented as men (Committee on the Participation of Women, 2003).

Students in several programs have described the importance of having “critical mass” of women or students of color (Cooper, 2000; Manzo, 1994). Graduate women in mathematics, computer science and physics have reported feeling isolated or alienated in their male-dominated departments, and have described ways that they feel that they do not fit in (Becker, 1990; Etzkowitz et al., 2000; Herzig, 2004b; Hollenshead et al., 1994). Women in mathematics in the U.S. have few female role models to guide them; in the fall of 2006, while 30% of full-time graduate students were women, only 12% of full-time doctoral faculty were women (Phipps et al., 2007).

Male science students have enhanced relationships with faculty compared with women, which provide men with increased opportunities to develop a sense of belonging. Henry Etzkowitz et al. (2000) argue that this feeling of acceptance is a prerequisite for independent and autonomous work. Denied the same degree of relationships with faculty, female students in science have a more difficult time acting independently. Further, women’s socialization may lead them to look for interaction and reinforcement, rather than to be autonomous and independent learners (Etzkowitz et al., 2000; Fennema & Peterson, 1985). This pattern of socialization can work against them in the eyes of their advisors, especially in a disciplinary culture like that found in mathematics, where work is expected to be individualistic and independent. Consequently, women graduate students in science and mathematics have been



stereotyped as less capable and uncompetitive, and as a result they often are not taken seriously by faculty (Becker, 1990; Committee on the Participation of Women, 2003; Etzkowitz et al., 2000; Stage & Maple, 1996). In this way, obstacles to developing feelings of belongingness are circular: women have more limited opportunities to develop a sense of belonging, which makes it more difficult for them to behave independently. The perception that they are dependent results in negative judgments of their abilities by faculty, which limits their further opportunities and makes it even more difficult for them to come to feel that they belong in mathematics.

A community of practice imposes certain cultural practices and implicit expectations on students (Lave and Wenger, 1991). The isolation, sexism, lack of role models, and stereotyped understandings of women's interactions in graduate school can combine to demonstrate to women ways that they do *not* belong in the male-dominated cultures of their departments and disciplines (Etzkowitz et al., 2000; Herzig, 2006; Hollenshead et al., 1994). In this sense, the communities of practice of graduate school and of mathematics may set expectations which some students are unable or unwilling to meet. It is therefore possible that people who persist in mathematics are those who are able or willing to adapt themselves to those cultural practices; that is, they learn, or are self-selected, to work within the existing structure, to play by the existing rules (Stage & Maple, 1996). In a study comparing the careers of women and men scientists, Sonnert and Holton (1995) found

little evidence that women in science follow or believe in a radically different epistemology or methodology that some feminist theorists of science have suggested. It may, of course, be proposed that women (and men) with alternative methodological and epistemological approaches do not flourish or survive in the science pipeline for very long, so that the scientists who are reasonably successful under the current system of science are predisposed to it, or at least have learned to accept it. (p. 156)

Individuals whose talents, values, skills, or interests make it difficult or undesirable for them to adapt to that structure may not be able to successfully negotiate the educational and professional systems that are necessary to allow them to do mathematics. In this study, I examine the social and cultural practices that are implicit in graduate mathematics education, and how those practices impact the experiences of women trying to become mathematicians.

#### *1.4 An Institutional Ethnography Approach*

Institutional ethnography is a method of investigation that explores how social settings

(in this case, mathematics graduate programs in the U.S.) are organized, and the web of social relationships that are the basis of those settings, through individual's experiences. This approach "attempt[s] to uncover, explore, and describe how people's everyday lives may be organized without their explicit awareness but still with their active involvement" (Campbell, & Gregor, 2004, p. 43). The focus of an institutional ethnography is to "learn to think, hear, and talk about the setting as various participants know it, but . . . also attend to . . . how a setting is organized" (Campbell & Gregor, p. 50). In this study, I attempt to use women's stories to reveal the social relations of mathematics graduate study, and how those relations organize women's experiences and opportunities within mathematics.

Institutional ethnography begins by locating a standpoint in an institutional order that provides the guiding perspective from which that order will be explored. It begins with some issues, concerns, or problems that are real for people and that are situated in their relationships to an institutional order. Their concerns are explicated by the researcher in talking with them and thus set the direction of inquiry. (Smith, 2005, p 32).

In this study, I begin "in the local actualities of the everyday world, with the concerns and perspectives of people located distinctively in the institutional process" (Smith, 2005, p. 24), with the perceptions of women graduate students. From their perspectives, I examine how their experiences are situated in the institutional order of graduate study in mathematics, with a focus on the opportunities and obstacles they face to developing a sense of belonging in mathematics.

The importance of belonging and the obstacles women face in developing a sense of belonging in graduate mathematics form the *problematic* (Campbell & Gregor, 2004) of the present study. I investigate this problematic through interviews with 12 women graduate students in mathematics in the U.S. As these women's experiences and stories unfold, we see the ways that the social order of graduate mathematics leads them to both seek and resist belonging in mathematics.

## 2. Method

This study is based on interviews with 12 women graduate students in mathematics, four enrolled in each of three PhD-granting mathematics departments at large, public universities in the U.S. These interviews were conducted as parts of larger and different studies, in which women and men graduate students and faculty were interviewed about their experiences in mathematics and beliefs about mathematics and graduate mathematics education. Two of the

three departments were among the 25 U.S. mathematics departments with the highest percentage of PhDs being earned by women over the period 1996-2002 (Jackson, 2004).

The women included in this analysis were selected as a purposeful sample from the larger samples from which they were drawn, using maximum variation sampling (Patton, 1990). They were selected order to represent as broad as possible a range of experiences and identities within and beyond mathematics. In this way, the relatively small sample size can be turned “. . . into a strength by applying the following logic: Any common patterns that emerge from great variation are of particular interest and value if capturing the core experiences and central, shared aspects of impacts of a program” (Patton, 1990, p. 172). Some participants were in their first year of graduate study at the time of their interviews, while others had been enrolled in graduate school for more than 6 years. They had received their undergraduate training at a range of institutions from many parts of the U.S., both public and private, both large and small. In all, their stories reflect their experiences in 15 mathematics departments as both undergraduate and graduate students. Most had entered graduate school directly after completing their undergraduate training, although several had either attended other graduate schools or had worked for between one and more than 10 years before entering their current graduate program. They ranged in age from their early 20s through their early 40s. Four of the women were African American and the remaining eight were White. Half of the women were married, two had young children, and two others discussed their plans to have children soon. One woman disclosed that she was a lesbian. Although all 12 women had entered the graduate program either intending to complete a PhD or considering the PhD as an option, by the time of their interviews, two of them had decided leave their programs after completing Masters’ degrees (one of the two had already left the program by the time of her interview); as of this writing, 4 of the 10 other women had completed the PhD, and 6 were progressing within their programs.

The structure of the interviews was largely the same at all three institutions. Participants were recruited by email, and asked to participate in an interview about their experiences in mathematics. All volunteers were given outlines of interview topics in advance of their interviews, and were encouraged to add things they thought were relevant and delete things they did not wish to discuss (after Burton, 1999; see Appendix I). Interviews covered participants’ mathematical “autobiographies”, their reasons for attending graduate school in mathematics, their interests and goals in mathematics, and their mathematical experiences both in and out of

school. The interviews were open-ended, progressing as unstructured conversations about participants' experiences, allowing them the opportunity to discuss "the web of feelings, attitudes, and values that give meaning to activities and events" (Anderson & Jack, 1991, p. 12) and to give them "the space and the permission to explore some of the deeper, more conflicted parts of their stories" (p. 13). Participants were encouraged to guide the conversation to those aspects of their experiences that they thought were most relevant. Consequently, not all interviews covered exactly the same topics.

To protect the women's anonymity, interviews were conducted in a private room on each campus outside of the Department of Mathematics. All interviews were tape recorded and the tapes were transcribed. Interviews ranged in length from thirty minutes to two and a half hours.

Transcripts of the interviews were analyzed inductively. Transcripts were read and re-read, and initial codes were developed to reflect what these women talked about concerning issues of belonging within mathematics, including obstacles they experienced. As coding progressed, new codes were developed and applied, and other codes were deleted or combined. Once the coding scheme reached a point at which it seemed to capture all relevant parts of the women's stories, an independent coder coded two interviews to check for reliability. There was a strong degree of agreement between the two coders, and any discrepancies were negotiated, resulting in several additions and clarifications to the coding scheme. Finally, all of the interviews were re-coded. The codes provide the organization for the results that follow.

Because of the small number of women interviewed and the ways that each interview was unique, statistical information about their responses is not provided. The narrative that follows weaves together issues that were common among the women's stories about belonging in mathematics, using the words of each of the women; discrepancies and contradictions are noted when they arose. Participant quotes have been edited for readability, and to obscure any personally-identifying information. All names are pseudonyms. In order to protect participants' anonymity, only limited information is provided about individual women.

### 3. Women's Experiences of Developing a Sense of Belonging

Through these open-ended interviews, the participants discussed three general themes concerning their experiences developing a sense of belonging in the social worlds of mathematics graduate study: the importance of having and being role models, the challenges they felt "fitting in," and their unwillingness or inability to focus on mathematics to the exclusion of

all else. Each of these themes is explored in subsequent sections.

### *3.1 Having and being role models*

Most of the women had people they identified as role models or mentors, including undergraduate or graduate professors, and for many, more advanced graduate students or recent graduates. Most of them looked to, or looked for, successful women mathematicians and graduate students who could help them see that they, too, could build a satisfying life in mathematics. For some, the small numbers of women mathematicians they had encountered as undergraduates or in graduate school left them feeling that there were not people around to whom they could relate.

If there was a good female role model that I felt like I could relate to that would really push me a lot better. . . . A female professor pushing me intellectually seems to be more what would actually drive me.

Seeing successful women in mathematics—particularly women who were mothers—helped them believe that they could succeed as well. Some women really appreciated role models who demonstrated the possibility of balancing family responsibilities with work as an academic. When I asked if there was anyone she would consider a mentor to her, one woman replied,

I look at balancing life and math. . . . The women in the department definitely because they've got their family and had kids. . . . So I look to them as people who are good at balancing. . . . I guess anybody who got a PhD would be somebody I look to, and they still have a life. Cuz I don't want to study all the time. . . . The faculty here, they didn't have kids until they were faculty themselves. Even Audrey, she had her first child during her postdoc. So [I feel I need] role models for having children in grad school and being a woman.

Each of the four Black women I interviewed (in two different departments) had heard about the three African American women who had earned PhD's from the Department of Mathematics at the University of Maryland in 2000 (Argetsinger, 2000)<sup>2</sup>. They each spoke of ways that those women's examples were important to them, as a way of proving that they too could achieve this.

I read in the paper of the three Black women that [the University of Maryland] had graduated in the PhD program. I thought that was phenomenal. I was like, “Wow, if they can do it, I can probably do it too.”

One woman explained that, while she felt her (white male) advisor was readily available to talk with her whenever she needed, she generally chose not to talk with him, because she did not feel that he could relate to her experience as an African American female.

Talking to African American females in mathematics is more personal. They understand. A lot of times before I could even get it out of my mouth, [they would know that], “this is the experience you’re having.” Dr. Smith won’t know that. Cuz he’s a man. He’s older. He’s Caucasian. He just won’t know that. . . . I don’t have time for generic [advice]. I can read it out of a book. . . . That’s not to say anything bad about him, it’s just who he is. He wouldn’t know.

One surprising issue that arose was the desire and expectation that some of these women felt to *serve* as role models for other women. This sentiment was expressed particularly eloquently by one Black woman, who felt motivated to achieve a PhD not just for herself, but for others who she felt she represented.

A PhD carries more weight for me as a black woman than it does for my [classmates]. I feel that getting a PhD is not about me. . . . For me, getting a PhD, that’s for me, that’s for my culture, my ethnicity, that’s for [my undergrad college].

This woman was very proud to work toward a PhD in mathematics for all that it might do to reflect positively on the preparation she had received at the historically black college she had attended, to which she felt intensely loyal. Another woman described her admiration for another graduate student whom had set a personal goal to become a role model for other young women in mathematics.

It would be cool to have a lot more female role models in Math. One of my friends who was here last year and who had transferred out of this program . . . She wanted a role model but she also felt she wanted a PhD in order to be a role model for other women and that was her driving force. Which I thought was kind of neat.

For some, this meant that they felt obligated to prove that women could achieve in

mathematics to the same extent as men.

When I was in college, I got pressure to go to grad school because I was a woman. . . . The problem with women not going to grad school and here you are good at math and you should go to grad school and not be a gender traitor or something like that . . . . To not go to grad school would have been lightweight, and just support the theory that women are lightweights when it comes to math.

Serving as a role model for other women, or for men about what women can achieve, was a double-edged sword for these women, placing substantial pressure on them to work to a higher standard than the male students in order to disaffirm stereotypes and prove what women could do.

The women I know that are looked at as knowledgeable people have to really, really prove themselves in order for them to gain the respect of peers and faculty. . . . I've seen examples [like] a male professor could not really handle women in the class, basically just dismiss them as incapable.

If you ask a question and it reveals your ignorance of the subject that you're studying, then you're the girl who doesn't know what goes on. That's different from being somebody who doesn't know what's going on.

Having role models, and being role models for others, helped these women identify others with whom they could affiliate in mathematics, and supported their beliefs that they could succeed. As I will discuss later, these affiliations helped these women construct a mathematical community in which they felt they belonged, countering some of the isolation they otherwise felt in their programs. However, these affiliations also carried burdens within the social relations that organize their graduate programs, as they felt pressure to prove their worth and invalidate negative stereotypes of mathematical women.

### *3.2 Fitting In*

As one woman described above, being some women felt that they stood out as different from the other students. There were a number of ways in which these women spoke about their challenges in feeling that they belonged in mathematics and in graduate school. For some women, being in a program with mostly men made them feel intense competition, and was sometimes intimidating. Most of the women graduate students described ways in which they felt

uncomfortable being in an environment with so few women.

I sometimes walk into a room, look around, realize I'm the only woman in the room, *again*, and it has an effect . . . . It makes me feel like on some level most of the people I interact with are missing one particular thing in common with me and I find that discouraging. . . . There are still sometimes times when it feels uncomfortable that there aren't more people like me.

One woman described how she felt that it was unacceptable to show femininity.

The math department [in graduate school] was not a very comfortable place for women. . . . I remember wearing a skirt and having people tease me endlessly about it. . . . "Oh, you're all dressed up today." "Hey, did you know you're wearing a skirt today?" And I felt like to be female, to show my femininity, was not acceptable.

The African American women felt doubly isolated, both as women in a male-dominated discipline and as Blacks in a largely White discipline.

I guess being in a room full of White people me acknowledging my Blackness, not that I'm always thinking about it but it's more aware than if I were in a room full of Black people. I'm just aware that there's no one else in the room who looks like me. And that kind of makes me, not nervous but it's kind of like, "That's strange, there should be someone else in here."

Each of the four African American women had attended an historically black college as an undergraduate, and they described ways in which the transition from a primarily Black institution to a primarily White institution entailed an adjustment for them. They felt that they stood out in some situations, and felt invisible in others. These adjustments were described as a combination of cultural difference, having moved from one part of the country to another, where humor, dietary habits, social expectations, and other habits and customs were different; and sometimes intolerance, describing some interactions with faculty that were blatantly racist.

When we were registering to start classes. . . . [T]hree of the other [African American] students had a couple different advisors who said, "The five of you are here under Affirmative Action so you probably should start with undergrad courses first so you can catch up with everybody else." . . . Another professor [told me], "Graduate school isn't



for everybody. Maybe you should consider something else.” But we had all gone there to visit at the same time and none of these comments were said at that time. There were three African American students there when we were going to visit but by the time we got there they were gone and they didn't let us talk to them, which should have been a heads-up that maybe something's wrong.

The African American women also described interactions with professors who “stiffened up” and were much less forthcoming with help and advice with African American students than with White students, or advised them to drop classes when they were struggling, rather than offering to help them learn. They also described their struggles to earn the respect of the undergraduate students they taught.

I think my color has something to do with it too, when I go in to teach the students, I guess they think I don't know as much as I do. They kind of try to second guess me. I can tell they would oppose me more than they would a White male or somebody like that, just because I'm a woman.

Another topic that some of the women discussed is the stereotype of mathematicians or mathematics students as being “nerds” or “uncool.” Mathematicians and mathematics students have commonly been stereotyped as lacking in social skills (Damarin, 2000; Campbell, 1995). Nel Noddings (1996) argues that

There seems to be something about [mathematics] or the way it is taught that attracts a significant number of young people with underdeveloped social skills. . . . If this impression of students who excel at math is inaccurate, researchers ought to produce evidence to dispel the notion, and teachers should help students to reject it. If it is true, math researchers and teachers should work even harder to make the “math crowd” more socially adept. Because that group so often tends to be exclusive, girls and minority youngsters may wonder whether they could ever be a part of it. But when the group is examined from a social perspective, many talented young people may question whether they *want* to be a part of it. (p. 611; italics in original)

One woman spoke at length of the lack of social skills among many of the graduate students in the department, calling them a “big collection of freaks.” Another woman disliked the way that she was stereotyped as a mathematics student, and felt that this was one reason why

younger students get disaffected from mathematics.

I think mathematics definitely has this stereotype which I really can't stand that it's dorky and I get made fun of by my non-mathematician friends a lot. . . . If somehow that could be changed, this is where I think that role models would come in. If people could look up to people who they thought were similar to them. . . . Especially in lower mathematics, [a lot of people] think math isn't cool and it's not interesting . . . and they don't see themselves doing it because of the stereotype. If they had role models that they felt were more like them, weren't dorky, then they would draw in a broader range of people. Instead of continuing this type of personality is mathematicians so everybody after that is only those types of personalities can become mathematicians.

Overall, these women did not feel that they fit in in mathematics. They felt uncomfortable in classes and other settings in which there were few women; the isolation was even more extreme for Black women or for women who were mothers. They felt distanced from mathematics by stereotypes of mathematical people as lacking in social skills. In each of these ways, these women faced explicit obstacles and clear messages about ways that they did not belong in mathematics.

### *3.3 Being unwilling or unable to become engrossed exclusively in mathematics*

Most of the women interviewed majored in mathematics in college because they realized that they were good at it, that it "came naturally" to them, or that mathematics was fun, but many of them explained that mathematics was only one of several interests they might have pursued. They chose graduate studies in mathematics for a range of reasons, including the desire to teach, a passion for learning more mathematics, and even "nothing better to do," many of them emphasizing the happenstance that led them there. This advanced graduate student, who is now an assistant professor of mathematics, summarized the feeling of many, emphasizing that while she was good in mathematics, it was only one of many things she was good at.

It was just something to do and something that I thought I would actually like doing. I don't think I had this burning drive to go to math grad school but I realized I could and I thought I would like it, and I didn't have anything else to do.

All of the women reported having some times when they experienced doubts about continuing with their graduate studies, wondering whether it was worth the sacrifices to their

personal lives that persistence would require. Like this first year student, who was also the mother of a young child, the women described some of the things they were giving up to pursue the PhD.

My friends are starting to work, starting to have their lives. They tell me they went to this concert, this show or play, and I want to go and do that stuff too. I just still have that drive to get that PhD . . . . I know it's hard, but I feel like in the long run it will be worth it.

The women's longing for a life outside of mathematics was balanced by what many of them described as sheer stubbornness that allowed them to persist in the program, despite the stress of exams, intense demands of coursework and teaching, and frustrations with research. Many women acknowledged that they had to have a love for mathematics in order to find the motivation to continue. However, the women also reported that they were not as focused on mathematics as they perceived the other students around them to be, as this first-year student explained:

I thought it was kind of strange that my classmates would talk about different math books like they would novelists like Toni Morrison or Faulkner. That's who I would talk about if I were talking about books [I had read]. I wouldn't talk about Rotman or math authors. I had to get used to that.

The woman who made the following statement had entered graduate school to earn a Ph.D., but at the time of her interview was preparing to leave after completing her Masters degree:

When you enter grad school you realize that there are people who are really, really interested in math. I kind of figured that I had enough interest in it to do it, but then you realize that there are people that spend their extra time doing it. That's what the program is made for, I feel, for people like that.

While these women generally felt they had the determination, will to work hard, and perseverance required to complete the PhD, they also placed limits around their commitment to mathematics, and they clearly identified themselves as being more than just mathematics students. They described their desire *not* to be totally identified with mathematics, and

emphasized that they belonged not only to mathematics, but were well-rounded individuals with other interests and obligations.

Their stories highlight a tension between being mathematics students and their need to fit in, through which their desire to be more than “only” mathematics students was confronted with a social order in which they felt that total devotion was required of them. For some, this disconnect between the way they perceived that others were fully absorbed in mathematics and their own sense of themselves as more “well-rounded” led them to question whether they fit in in the social and academic worlds of graduate school, and presented an intrinsic obstacle their development of a sense of belonging in their graduate programs.

Suzanne Damarin (2000) compares people with mathematical ability to other “marked categories” such as women, people of color, criminals, people of disability, or homosexuals, and identifies these characteristics of marked categories:

1. Members of marked categories are ridiculed and maligned, and descriptions of marked categories are used to harass, tease, and discipline members of the larger society.
2. Members of marked categories are portrayed as incompetent in dealing with daily life.
3. In institutions designed to meet the needs of all, the needs of members of marked categories are deferred, compared with the needs of the unmarked.
4. Members of marked categories are feared as powerful even as they are marked as powerless.
5. Marking serves to define communities of the marked.
6. Membership in multiple marked categories places individuals in the margins of each marked community.
7. The study of a marked category leads to the construction and study of the complementary class of people.
8. The unmarked category is generally larger than the marked category; even when this is not the case, the marked category is not recognized as the majority. (Damarin, 2000, pp. 72-74)

Damarin then presents an analysis of discourses surrounding mathematical ability, and concludes,

From leading journals of public intellectual discussion, from the analyses of sociologists of science, from the work of (genetic) scientists themselves, from the pages of daily

papers, and from practices of students and adults within the wall[s] of our schools, there emerges and coalesces a discourse of mathematics ability as marking a form of deviance and the mathematically able as a category marked by the signs of this deviance. (p. 78)

If students in advanced mathematics are indeed marked as “deviant” because of their mathematical talent, women students are marked within this group, and may suffer the double stigma of not being “real” mathematicians because of their gender, and not being “real” women because of their work in mathematics. Given the common perceptions of mathematics students as being white, male, childless, without interests outside of mathematics, and socially-inept, it may be that members of various groups recognize tangible ways in which they do *not* fit in with this group, and do not *wish* to fit in. Thus for some students who already feel marginalized in some communities, belonging in mathematics may not be an entirely good thing: while belonging facilitates persistence and success in mathematics, it also “marks” a student as deviant, as socially inept. Women who choose to pursue mathematics must be willing to endure these multiple constructions of themselves as deviants, both as women and as mathematically competent. The women in this study described ways that they worked to distance themselves from some of these common constructions of mathematical deviance, which, paradoxically, led them to resist belonging in mathematics.

These women all struggled in various ways to balance their lives, obligations, and identities in and out of graduate school. These issues of balance took different forms depending on the women’s life circumstances, but they generally described the conflicts they experienced in building a well-rounded life that included both graduate school and other commitments and interests.

Most of the women appreciated the flexibility of life as a graduate student, where they were mostly responsible to themselves for managing their own time. Of course, this flexibility can backfire, as it also meant that there were times when they could fall far behind in their studies. Several of them felt that they were progressing through graduate school more slowly than the “norm” or than their advisors expected, because of decisions they had made not to “bury myself in my mathematical life to the exclusion of all else.” One fourth-year student lamented how competitive she perceived graduate school to be; she hated the emphasis she heard repeatedly on how long it took various students to earn their degrees and the pressure that imposed on her.

None of them objected the hard work, and they acknowledged the richness of their learning that came from immersing themselves in mathematics, balanced with the opportunities to learn to teach. This woman described how the satisfaction of learning and focusing on her own progress worked against these frustrations:

I'm learning all this stuff. If I work hard enough I can learn it. And I have. I do feel like I belong. Even though I struggle and sometimes say I don't.

This woman had had very difficult struggles in her family life, including health challenges, which interfered with her ability to engage fully in her studies for some periods of time and made it difficult for her to engage in the social and academic worlds of her program. However, she persisted, has progressed in her program, and hopes to finish soon. Like many of the women, balancing their lives as graduate students with their other responsibilities was a significant challenge, and she expressed her frustration at not being able to devote herself fully to either her family life or studies.

Earlier, I discussed the women's perceptions that other graduate students were totally absorbed in mathematics. The women I interviewed struggled to find a place for their non-mathematical selves in their lives, *insisting* on having a life outside of graduate school, refusing to let mathematics become all that there was in their lives. They spoke about the need to take care of themselves and to have other interests, including time for rest and exercise, family and friends, participating in church communities, taking courses in other subject areas, playing music, volunteer work, and dating—all the things that make up the lives of a varied group of people. Some of the women who were mothers described ways that their professors and advisors supported their need to meet family responsibilities. But others also felt a sense of disapproval from their advisors and professors for having outside obligations. This fourth-year student was studying a world language in order to connect herself more strongly to her ethnic heritage, and was committed to volunteer work she did in her community:

Sometimes I feel like I'm disapproved of for having a life outside of math. This whole idea that, why should I be wasting time doing anything else when I could be doing math? No wonder you're failing the qual. That type of thing. That really bothers me. I should be allowed to have a life outside of math. This whole attitude that I think a lot of mathematicians have that if you do math, you do it because you love and it's all you want

to do every day, all day. Well no, that's not actually true for me. I like to do it sometimes but I like other things.

In addition to describing these outside activities as essential parts of their lives that they valued and enjoyed, these activities represented explicit strategies to find time away from school, to help them “escape” or get a break from thinking and talking about mathematics. Some spoke of the need to manage frustration and stress by getting away from campus for a while, or felt it was critical to have friends or partners who are not in mathematics, so that they could escape from mathematics for a time and talk about other things. They had other explicit strategies for getting time away from their studies, including never studying on a Saturday night, volunteer work, teaching exercise classes, or positions in church governance. Many of these women highly valued time spent with their extended families and church communities, and they were often frustrated by how difficult it was to find the time to do this, as even a Saturday afternoon off could leave them feeling hopelessly behind in their studies. For example, this third-year student had gone through some difficult challenges in her personal life. She cried as she said, “I just feel that I want to do everything and I just get frustrated that I’m mortal and I can’t.” Paradoxically, she went on to describe school as an escape from those stresses.

I think school's a nice escape from it. As long as I'm not talking with my officemates who are my friends, it really doesn't come up so much, which is nice.

This African American woman was the mother of a young child, and described how important it was to her, when she came to visit her graduate school before deciding to enroll there, that

It seemed like everybody here has outside lives. They're not so consumed in doing their mathematical work. They actually have families. A lot of people here are couples and married, have children, and by me having a child and a husband I felt it was more suited toward me.

Probably like most working mothers, the graduate students who were mothers experienced a double-bind when it came to balancing motherhood with graduate school. Their family responsibilities left them feeling that they did not have sufficient time to devote to their schoolwork, like this mother of two young children:

I always envisioned getting your PhD is like preparing for an Olympic sport. You really have to throw 110% of yourself into that. And I have other obligations. I can't give 110% of myself to this goal.

Conversely, when devoting time to schoolwork, they felt pressure, stress, and guilt over not spending more time with their children, like this mother:

I don't get to see her that much. It makes me sad because she's a child and she needs her mother, but in the long run it will be more helpful to her . . . This is how I look at it. I don't remember what happened to me before I was 5 so hopefully she won't either. But we give her lots of love.

While she imagined that other graduate students might have time in the evenings to themselves, the little time she had away from her schoolwork was devoted to caring for her daughter, so she had little time for herself, social engagements, or anything outside of her studies and her daughter. The need to work as a teaching assistant added to this pressure.

Graduate study is a "greedy institution" (Coser, 1974; cited in Grant, Kennelly, & Ward, 2000), and as such demands undivided loyalty and "total commitment from participants and the relinquishing of competing commitments" (Grant et al., 2000, p. 63). These women described an unwillingness to devote themselves to mathematics "110 percent," and ambivalence about whether or not they *wanted* to belong in mathematics, which represents an intrinsic obstacle to belonging in mathematics.

At least two of the women were considering having children soon, and understood how difficult it might be to balance the demands of a family and school. They each had seen other students who were parents, and they observed the challenges involved in parenting while in graduate school, including the financial pressures of having a child on the limited income of a graduate student. One fourth-year student, who was beginning to work on her dissertation, said:

We plan to have children soon, and how are we going to save money and buy all the things a baby needs? . . . I couldn't imagine doing this with kids. . . . I couldn't imagine doing it with a family, I think it would be so difficult.

One of the graduate students who was a mother repeated the advice that she had once heard:

Choose the right advisor. If your advisor accepts that your kid is your first priority and



your degree is your second priority, you're going to have a much easier life than if your advisor doesn't acknowledge the existence of your family.

Her advisor fully accepted her obligations as a parent, inviting her to bring her child along to meetings or re-scheduling an exam when she could not find childcare. At the same time, the department insisted that she be enrolled in a full-time load of 9 credits in order to receive financial support. Several mothers described how this course load, coupled with teaching responsibilities, posed a significant challenge to them as parents.

Parenthood is another greedy institution, particularly for women (Coser, 1974; cited in Grant et al., 2000). The conflict between the two greedy institutions of motherhood and graduate school can be substantial. Of course, while some fathers are involved in child care and male students may also experience conflicts between school and parenting, women experience the additional pressure about the concurrent timing of graduate school and their childbearing years. Graduate school is not structured to accommodate childbearing and childrearing demands, and family responsibilities affect women graduate students more strongly than men (Lovitts, 2001; Nerad & Cerny, 1993; Sonnert & Holton, 1995). Women graduate students in science who marry or have children have been viewed as not serious about their studies, or as unreliable and not worth the investment; men who marry or have families do not face the same biases (Etzkowitz et al., 2000). In this sense, women who are both mothers and graduate students are assumed to have conflicting loyalties, and are marked as not serious students. In mathematics in particular, some women have reported having left graduate mathematics altogether due to the perceived incompatibility of the life of a doctoral student in mathematics and a personal life outside of mathematics (Stage & Maple, 1996). The women interviewed in the present study describe the high costs of considering parenting for those women who chose to remain, and the high cost of graduate study for those who are parents.

Students are members of a range of communities of practice, including school, family, and other communities. For graduate students who are also parents, or who have other commitments or interests outside of school, the conflicting demands of time, energy, and attention can serve to make it more difficult for them to become integrated in the mathematical communities of graduate school. Of course, many of these obstacles may affect both women and men; what is noteworthy about these women's experiences—even if not unique to them—are the intentional choices they described to distance themselves from mathematics, despite their passion

and dedication to their studies.

#### 4. Conclusion

The need to belong is perhaps among the most fundamental human needs (Flinders, 2002). Carol Lee Flinders (2002) searched extensive anthropological and archeological records, and provided an historical analysis to argue that prior to humankind's mastery of agriculture ten thousand years ago, what she calls the "values of belonging"—such values as cooperation, intuition, balance, deliberateness, mutuality, affinity for alternative ways of knowing, and inclusiveness—were fundamental to the organization of many cultures around the globe. For foraging peoples, there was no motivation to compete, to exclude, or to acquire; instead, the values of belonging were the foundation of social organization. But, with the advent of agriculture, a new culture of acquisition and competition began to develop, and social divisions based on status and domination evolved. Cooperation and belonging became a lower-order priority, as the notion of privilege itself became privileged;

As I argued earlier, success in graduate school necessitates learning mathematical content, participating in mathematical practices, and coming to belong in mathematics (Boaler, 2000; Herzig, 2004a). It has been argued that students' integration into the academic and social communities of their departments and programs is critical for their persistence in graduate study; further supporting the importance of developing a sense of belonging in graduate mathematics. While graduate mathematical education has long emphasized the teaching of mathematical knowledge, and increased calls have recently been made to train graduate students in a range of mathematical practices (Bass, 2003), students' coming to feel that they belong in mathematics has been largely unexamined.

I documented some obstacles to belonging faced by female graduate students in particular. Many of these obstacles are not surprising: difficulty in identifying role models; the burden of having to prove their worth and the worth of all women in mathematics; conflicting demands of family and school obligations, particularly the demands of childbearing and childrearing; and the isolation of life in a (mostly White) male-dominated discipline. These obstacles may help explain, at least in part, the small numbers of women entering graduate school and completing the PhD in mathematics.

Damarin (2000) argues that membership in the deviant category provides the "deviant" with a community with which to affiliate; being identified and marked as mathematically able

allows the mathematics graduate students to form a community among themselves. Unfortunately, women are members of (at least) two marked categories, and the double marking is not simply additive; that is, it is not the case that they simply belong to a separate marked category of “mathematically able women.” Instead, they are constructed as deviant separately within each marked category. First, they are marked as women, but among women, their mathematical ability defines them as deviant. Second, given common stereotypes of mathematics as a male domain, mathematical women are marked among mathematicians as not really one of them. For women of color, the marking is three-fold and even more complex, leading them to be distanced from each of those communities to which they might otherwise belong. Women graduate students who are parents also suffer the multiple markings of being mathematically talented, being women, and being parents, and need to develop strategies to cope with these conflicting labels and their demands. Consequently, mathematical women do not have access to the mainstream community of the mathematically able, as their multiple markings marginalize them from this community.

Instead, women who elect to pursue mathematics are sometimes members of smaller communities which respect and reward mathematical abilities, partly countering the discourses that label their mathematical abilities as deviant (Damarin, 2000). Students in several doctoral programs have reported the importance of having a “critical mass” of women or students of color (Cooper, 2000; Manzo, 1994). The importance of role models and the presence of other women students to the women I interviewed may represent their attempts to build communities that affirm them as women and as mathematics students, since they do not have access to other mathematical communities that might serve this purpose. Without these smaller communities, women are left without a sense that they belong somewhere, *anywhere* within the world of mathematics.

Graduate mathematics educators need to question whether it is necessary to give oneself over to mathematics entirely, or if it is possible to do quality work in mathematics without this total devotion. In the famous remarks made by former Harvard President Lawrence Summers, he claimed that

the most prestigious activities in our society expect of people who are going to rise to leadership positions near total commitments to their work. They expect a large number of hours in the office, they expect a flexibility of schedules to respond to contingency, they

expect a continuity of effort through the life cycle, and they expect-and this is harder to measure-but they expect that the mind is always working on the problems that are in the job, even when the job is not taking place (Summers, 2006).

Despite these expectations, it may not actually be the case that this “near total commitment to their work” is necessary to enable productive contributions to scholarship. As these women described, when they are recognized and respected as complete human beings with interests and commitments outside of school, their opportunities to pursue mathematics are enhanced. At least some of these women might make valuable contributions to mathematics, and might be mathematically successful in other ways, even without devoting themselves to mathematics “110 percent.” At American University, where women of color were particularly successful, a commitment was made to “accommodating the busy professional and personal lives of the women, many of whom are working mothers” (Manzo, 1994, p. 40); students and graduates of that program reported that such flexibility was a critical factor in their persistence. Leaders in graduate mathematics education need to consider whether graduate programs might be re-conceived to accommodate the full and busy lives of students with a more diverse set of commitments and identities.

The institutional ethnographic approach used in this study used the reported experiences of 12 women to illuminate aspects of the institutional and social relations of graduate mathematics that present the women with an important and difficult tension. On the one hand, they described the myriad ways that they were socially isolated from other students around them and consequently had to struggle to find ways in which they could belong in mathematics. On the other hand, they described ways in which stereotypes of mathematics students and the multiple constructions of themselves as deviant as women, as mathematicians, and for some, as parents or people of color, led them to *choose* not to identify with mathematics.

Enhancing the diversity of mathematics graduate students requires a focused effort to build avenues for women and people of color to connect with the communities within their programs and departments, to develop communities, and to develop a sense that their mathematical abilities, their gender, and other aspects of who they are, are not deviant. This requires more than just bringing women and people of color into existing mathematics communities, but also requires a re-consideration of prevailing stereotypes and conventions among those involved in mathematics. As Noddings (1996) argued, mathematics educators need

to find ways to make the social world of mathematics more accessible to a broader range of people. Only then can women and some other groups of students come to feel that they truly belong in some part of the mathematical world.

## Notes

I would like to express my sincere gratitude to the 12 women whose stories are reflected here, for their generosity in welcoming me into their mathematical worlds and taking the time to discuss their stories with me. I would also like to thank Diane Gusa for her careful and thoughtful assistance in the data analysis.

This material is based upon work supported by the National Science Foundation (United States) under Grant No. 0346556. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

1. These three women were the first African American women to earn the PhD in Mathematics from the University of Maryland, and were half of the only six African American women who earned PhDs in the mathematical sciences *nationally* in that year. Their ground-breaking accomplishments were covered in the national media (e.g. Artsinger, 2000).

## References

- Allestaht-Snyder, Martha & Hart, Laurie E. (2001). "Mathematics for All": How do we get there? *Theory into Practice*, 40(2), 93-101.
- Anderson, Kathryn., & Jack, Dana C. (1991). Learning to listen: Interview techniques and analyses. In Sherna Berger Gluck & Daphne Patai (Eds.), *Women's Words: The Feminist Practice of Oral History*. New York: Routledge.
- Argetsinger, Amy. (December 22, 2000). Black Women beginning to count in mathematics. *The Washington Post*, p. B01.
- Bair, Carolyn Richert & Haworth, Jennifer Grant (1999, November 20, 1999). *Doctoral Student Attrition and Persistence: A Meta-Synthesis of Research*. Paper presented at the The Annual Meeting of the Association for the Study of Higher Education (ASHE), San Antonio, TX.
- Bass, Hyman (2003). The Carnegie Initiative on the Doctorate: The case of mathematics. *Notices of the American Mathematical Society*, 50(7), 767-776.
- Becker, Joanne Rossi (1990). Graduate education in the mathematical sciences: Factors influencing women and men. In Leone Burton (Ed.), *Gender and Mathematics: An International Perspective* (pp. 119-130). London: Cassell Educational Limited.
- Boaler, Jo (2000). Introduction: Intricacies of knowledge, practice, and theory. In Jo Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning* (pp. 1-17). Westport, CT: Ablex.
- Boaler, Jo (2002). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. *For the Learning of Mathematics*, 22(1), 42-47.
- Boaler, Jo, Wiliam, Dylan & Zevenbergen, Robin (2000). The construction of identity in secondary mathematics education. In J.F. Matos & M. Santos (Eds.), *Proceedings of the Mathematics and Society Conference*, Montechoro, Portugal: Investigação em Educação da Faculdade de Ciências Universidade de Lisboa. pp. 192-202.
- Burton, Leone (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, 37, 121-143.
- Campbell, Marie & Gregor, Frances (2004). *Mapping Social Relations: A Primer in Doing Institutional Ethnography*. Walnut Creek, CA: AltaMira Press.
- Campbell, Patricia B. (1995). Redefining the 'girl problem' in mathematics. In Walter G.

- Secada, Elizabeth Fennema & Lisa Bird Adajian (Eds.), *New directions for equity in mathematics education* (pp. 225-241). New York: Cambridge University Press.
- Committee on the Participation of Women (2003, January). Improving the Persistence of Women in Graduate Mathematics, *Symposium presented at the Annual Joint Meeting of the American Mathematical Society and the Mathematical Association of America*. Baltimore, MD.
- Cooper, Duane A. (2000). Changing the faces of mathematics PhDs: What we are learning at the University of Maryland. In Marilyn E. Strutchens, Martin L. Johnson & William F. Tate (Eds.), *Changing the Faces of Mathematics: Perspectives on African Americans* (pp. 179-192). Reston, VA: National Council of Teachers of Mathematics.
- Damarin, Suzanne K. (2000). The mathematically able as a marked category. *Gender and Education*, 12(1), 69-85.
- Etzkowitz, Henry, Kemelgor, Carol, Neuschatz, Michael, & Uzzi, Brian (1992). Athena unbound: Barriers to women in academic science and engineering. *Science and Public Policy*, 19(3), 157-179.
- Etzkowitz, Henry, Kemelgor, Carol, & Uzzi, Brian (2000). *Athena unbound: The advancement of women in science and technology* (Vol. 19). Cambridge: Cambridge University Press.
- Fennema, Elizabeth & Peterson, Penelope (1985). Autonomous learning behavior: A possible explanation of gender-related differences in mathematics. In Louise C. Wilkinson & Cora B. Marrett (Eds.), *Gender Influences in Classroom Interaction* (pp. 17-35). Madison: University of Wisconsin.
- Flinders, Carol Lee (2002). *The Values of Belonging: Rediscovering Balance, Mutuality, Intuition, and Wholeness in a Competitive World*. San Francisco: HarperCollins.
- Gerholm, Tomas (1990). On tacit knowledge in academia. *European Journal of Education*, 25(3), 263-271.
- Girves, Jean E. & Wemmerus, Virginia (1988). Developing models of graduate student degree progress. *Journal of Higher Education*, 59(2), 163-189.
- Golde, Christine M. (1996). *How Departmental Contextual Factors Shape Doctoral Student Attrition*. Unpublished Dissertation, Stanford University, Stanford.
- Grant, Linda, Kennelly, Ivy, & Ward, Kathryn B. (2000). Revisiting the gender, marriage, and parenthood puzzle in scientific careers. *Women's Studies Quarterly*, 28(1 & 2), 62-85.

- Herzig, Abbe H. (2002). Where have all the students gone? Participation of doctoral students in authentic mathematical activity as a necessary condition for persistence toward the Ph.D. *Educational Studies in Mathematics*, 50(2), 177-212.
- Herzig, Abbe H. (2004a). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research*, 74(2), 171-214.
- Herzig, Abbe H. (2004b). "Slaughtering this beautiful math": Graduate women choosing and leaving mathematics. *Gender and Education*, 16(3), 379-395.
- Herzig, Abbe H. (2006). "I just didn't see that I had a place in the old white guy's club": How can women and students of color come to *belong* in graduate mathematics? In Jill M. Byzinstinsky & Sharon R. Bird (Eds.). *Removing Barriers: Women in Academic Science, Technology, Engineering and Mathematics*. Bloomington: Indiana University Press. pp. 254-270.
- Hollenshead, Carol, Younce, Patricia Soellner, & Wenzel, Stacy A. (1994). Women graduate students in mathematics and physics: Reflections on success. *Journal of Women and Minorities in Science and Engineering*, 1, 63-88.
- Jackson, Allyn (2004). Has the women-in-mathematics problem been solved? *Notices of the American Mathematical Society*, 51(7), 776-783.
- Ladson-Billings, Gloria (1997). It doesn't add up: African American students' mathematics achievement. *Journal for Research in Mathematics Education*, 28(6), 697-708.
- Lave, Jean & Wenger, Etienne (1991). *Situated Learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Lovitts, Barbara E. (2001). *Leaving the Ivory Tower: The Causes and Consequences of Departure from Doctoral Study*. Lanham, MD: Rowman and Littlefield.
- Manzo, Kathleen Kennedy (1994). American University: Success is in the numbers, African American women excel in math Ph.D. program. *Black Issues in Higher Education*, 11, 40-43.
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Research Council. (1992). *Educating Mathematical Scientists: Doctoral Study and the Postdoctoral experience in the United States*. Washington, DC: National Academy Press.
- National Science Foundation. (1998). *Summary of Workshop on Graduate Student Attrition* (No.



- NSF 99-314). Arlington VA: National Science Foundation, Division of Science Resource Studies.
- National Science Foundation. (2008). *Science and Engineering Indicators 2008, Volume 2 (NSB-08-01A)*. Arlington, VA: National Science Foundation.
- Nerad, Maresi & Cerny, Joseph (1993). From facts to action: Expanding the graduate division's educational role. *New Directions for Institutional Research*(80), 27-39.
- Noddings, Nel (1996). Equity and mathematics: Not a simple issue. *Journal for Research in Mathematics Education*, 27(5), 609-615.
- Patton, Michael Q. (1990). *Qualitative Evaluation and Research Methods (2<sup>nd</sup> ed)*. Newbury Park, CA: Sage Publications.
- Phillips, Polly, Maxwell, James W. and Rose, Colleen A. (2007). 2006 Annual Survey of the mathematical sciences in the United States (Third report). *Notices of the American Mathematical Society*, 54(10), 1333-1344.
- Rogoff, Barbara (1994). Developing understanding of the idea of communities of learners. *Mind, Culture, and Activity*, 1(4), 209-229.
- Smith, Dorothy (2005). *Institutional Ethnography: A Sociology for People*. Lanham, MD: AltaMira Press.
- Sonnert, Gerhard & Holton, Gerhard (1995). *Who Succeeds in Science? The Gender Dimension*. New Brunswick, NJ: Rutgers University Press.
- Stage, Francis K. & Maple, Sue A. (1996). Incompatible goals: Narratives of graduate women in the mathematics pipeline. *American Educational Research Journal*, 33(1), 23-51.
- Summers, Lawrence (2005). <http://www.president.harvard.edu/speeches/2005/nber.html>.
- Tate, William F. (1995). Returning to the root: A culturally relevant approach to mathematics pedagogy. *Theory into Practice*, 34(3), 166-173.
- Tinto, Vincent (1993). *Leaving College: Rethinking the Causes and Cures of Student Attrition* (2nd ed.). Chicago: University of Chicago Press.
- Wenger, Etienne (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.

## **Appendix I**

### **Graduate Student Interview Outline**

The questions below are intended to give a direction to our discussion, but are not requirements for how it will develop. Feel free to delete anything from the list that you do not wish to discuss, and to add anything else that you feel might be relevant.

#### **About your interests in mathematics:**

- When did you first become interested in math? How did that interest develop?
- Have there been any people who have been influential to you in mathematics?
- What experiences have you had with mathematics, either in or out of school? How did those experiences affect you?
- Do you feel successful in mathematics?
- Why did you decide to come to graduate school? Why at [name of university]?
- What did you initially hope to do with the degree?

#### **About your experiences in graduate school:**

- Which aspects of graduate school met your expectations? Which didn't? Why?
- What have you enjoyed most about your experience here? What have you enjoyed least?
- What are the most important things you have learned? How did you learn those things?
- What relationships do you have with other students? With faculty?
- How have you learned (or are you learning) to do mathematical research, and to work as a mathematician?
- Are you involved in the department or the broader mathematical community outside of class?
- What does it take to succeed in graduate school?

#### **About your current goals in mathematics:**

- What are your current goals in mathematics?
- Did you ever have doubts about continuing? When did you first start having doubts? Why?
- Which factors have been the most helpful in helping you to stay in the program, and to succeed to this point?

**If you left the program, plan to leave, or are thinking of leaving without finishing:**

- At what point in your program were you when left or will you be when you leave?
- If you could change anything about the math department or program here to make it a better experience for you, what would you change? Are there things the faculty could have done to have made you more likely to stay?
- Do you have any second thoughts?
- What will you do/are you doing after graduate school?

## Magic Circles in the *Arbelos*

Christer Bergsten<sup>1</sup>

Department of Mathematics, Linköpings Universitet, Sweden

**Abstract.** *In the arbelos three simple circles are constructed on which the tangency points for three circle chains, all with Archimedes' circle as a common starting point, are situated. In relation to this setting, some algebraic formulae and remarks are presented. The development of the ideas and the relations that were "discovered" were strongly mediated by the use of dynamic geometry software.*

**Key words:** Geometry, circle chains, arbelos, inversion

**Mathematical Subject Classification:** Primary: 51N20

### Introduction

The *arbelos*<sup>2</sup>, or 'the shoemaker's knife', studied already by Archimedes, is a configuration of three tangent circles, and may indeed, due to its many remarkable properties, be called three magic circles. In this paper the focus will be on three circles linked to the *arbelos*, involved in the construction of Archimedes' circles<sup>3</sup> and chains of tangent circles. These three circles, which I also want to call 'magic', are displayed in the figure below. The *arbelos* is constructed by the circles with diameters BC, BF, and FC, respectively, and FG is the perpendicular to BC. The magic circles I will discuss in this paper are the circles with centre at B passing F, with centre at C passing G, and with centre on BC (at H) passing B and the intersection point R of the previous two magic circles. A parallel "mirror" construction of these circles can be performed on the right side of the segment FG.<sup>4</sup>

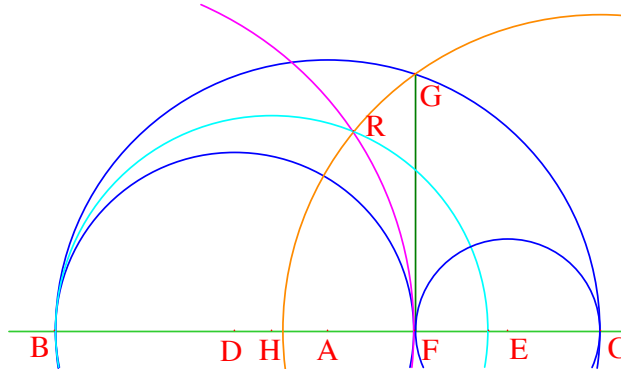
---

<sup>1</sup> Department of Mathematics, Linköpings Universitet,  
SE 58183 Linköping, Sweden; [christer.bergsten@liu.se](mailto:christer.bergsten@liu.se)

<sup>2</sup> See for example [3]

<sup>3</sup> See for example [3].

<sup>4</sup> After the publication of this paper as a pre-print [5], some of these circles were analysed, from a different perspective than in the present paper, as "midcircles" in [6].



As one starting point of my discussion, I will look at the following basic construction. One aspect of Archimedes' "twin" circles in the *arbelos* involves constructing a circle tangent to a given circle and its tangent. Given the point of tangency on the given circle this is easily done as in figure 1.

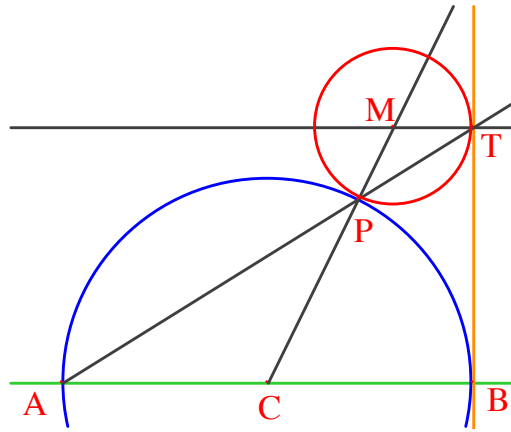


Figure 1. Construction of tangent circle

A circle on diameter  $AB$  with centre  $C$  is given<sup>5</sup>, as well as a point  $P$  on this circle, and the line tangent to the circle at  $B$ . Draw the line by  $A$  and  $P$ , meeting the tangent in  $T$ . The perpendicular to the tangent by  $T$  meets the line by  $C$  and  $P$  at  $M$ . Thus, the tangent circle is found with centre at  $M$  and tangency points  $P$  and  $T$ . The construction is built on the classic result that the segment connecting opposite endpoints of parallel diameters to externally tangent circles passes the point of tangency. That  $MP$  and  $MT$  are equal follows from the similarity of the triangles  $ACP$  and  $TMP$ ,  $AC$  and  $TM$  being parallel.

### Magic circle 1

It is evident that the circle  $C_{A,B}$  meets any such tangent circle at a right angle. To prove this, apply circle inversion of these three circles and the tangent line to the circle  $C_{B,A}$ , as in figure 2.

<sup>5</sup> For this paper I will use the notation  $C_{C,B}$  for a circle with centre at the point  $C$  with radius  $CB$ .

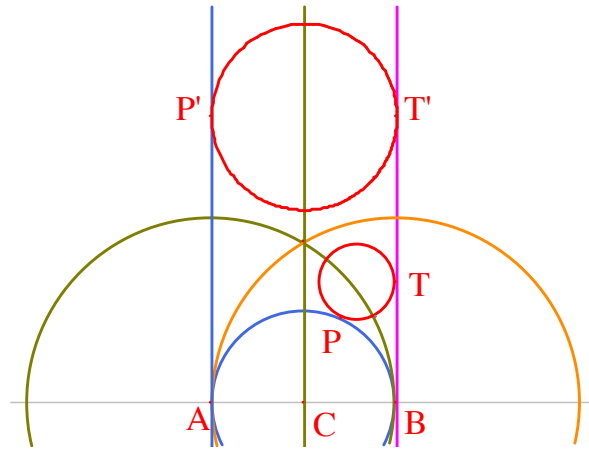


Figure 2. Inversion in circle with centre at B.

As point B is mapped, by this inversion, to infinity, the circle  $C_{C,B}$  is mapped to line through A and P', and circle  $C_{A,B}$  to the perpendicular to AB at C. P and T are mapped to P' and T', respectively, and the circle with tangent points P and T to the circle tangent to the lines by A and P', and T and T', respectively, thus orthogonal to the perpendicular to AB at C. Because of this orthogonality property I call the circle  $C_{A,B}$  a *magic circle*. How this circle is related to a circle chain in the *arbelos* will be discussed in the next section.

## Magic circle 2

Given its tangency points on the inner circles of the *arbelos*, the construction in figure 1 can be used to construct Archimedes' circles. A simple construction is shown in figure 3, where the circle  $C_{C,G}$  meets the left inner circle at point P, which is a tangency point of the left twin circle. Because of this I call the circle  $C_{C,G}$  a magic circle. The other two tangency points, P' and P'', are found by drawing the lines by B and P, and by C and P', respectively (see figure 3).

That P is a tangency point can be established algebraically, using the centre coordinates  $\left(\frac{r}{2}(1+r), r\sqrt{1-r}\right)$  of the left Archimedes' circle<sup>6</sup> to find the coordinates of its tangency point  $\left(\frac{r}{2-r}, \frac{r\sqrt{1-r}}{2-r}\right)$ , which has the same distance  $\sqrt{1-r}$  to the point C as the point G.

In addition to the properties shown in figure 1, the diameter at P' to the circle passing P, P', and P'', is parallel to the diameter BC of the circle  $C_{A,C}$ , showing that P'' is the point of tangency between these two circles as it is constructed by the line by CP', i.e. passing the end points of the parallel diameters.

<sup>6</sup> Here, BC = 1 and BF = r, with  $0 < r < 1$ . See for example [3].

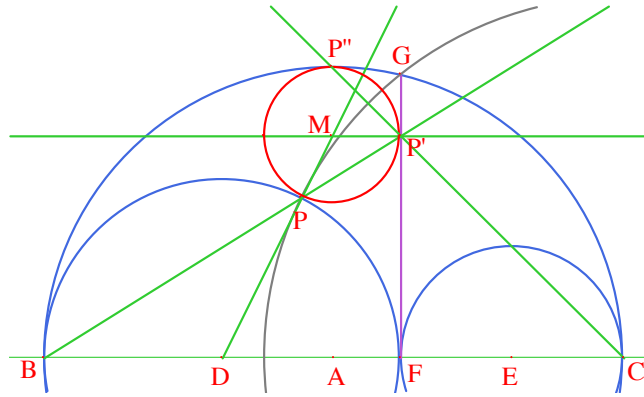


Figure 3. A construction of the left twin circle.

The centre  $M$  of the twin circle is found as in figure 1, where the perpendicular line to  $FG$  at  $P'$  meets the line by  $D$  and  $P$  (for this construction the point  $P''$  is not needed). The construction of the right twin circle is done the same way, by first drawing the circle  $C_{B,G}$ .

Applying the result from the inversion described in figure 2, chains of tangent circles as shown in figure 4 are easily constructed by drawing the magic circle  $C_{B,F}$  (and  $C_{C,F}$ ) in the *arbelos* (with notations as in figure 3)<sup>7</sup>.

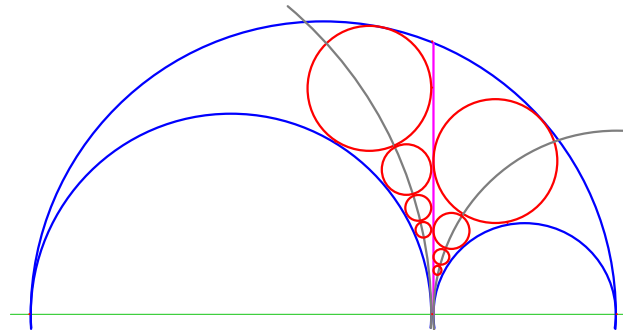


Figure 4. Circle chains in the arbelos

The mode of construction is the same as in figure 1, illustrated in figure 5, where the first circle in the chain after the left twin circle is shown. The inversion in figure 2 proves the tangency of this circle also to the left inner circle of the *arbelos*.

<sup>7</sup> This magic circle  $C_{B,F}$  is an example of a Woo circle (see [2])

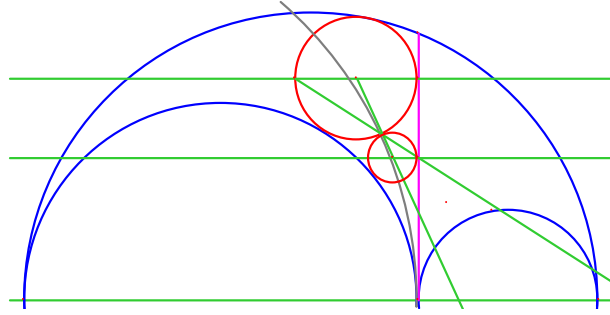


Figure 5. Construction of first tangent circle in chain

The radii of these chain circles can be evaluated (by algebraic calculation using Pythagoras' theorem on the marked triangles in figure 6) by the following recursion formula:

$$r_{n+1} = \frac{r \cdot r_n}{(\sqrt{r} + \sqrt{2r_n})^2}, \quad n=1,2,3,\dots, \quad \text{where } r_1 = \frac{r}{2}(1-r) \text{ is the radius for Archimedes'}$$

circle, as well known. By a straightforward mathematical induction, the explicit formula

$$r_n = \frac{r(1-r)}{2(1+(n-1)\sqrt{1-r})^2}, \quad n=1,2,3,\dots \quad \text{can be established. A calculation of the}$$

coordinates  $(x_n, y_n)$  of the circle centres gives

$$\begin{cases} x_n = r - r_n \\ y_n = \sqrt{2r \cdot r_n} \end{cases}, \quad n=1,2,3,\dots, \quad \text{where the y-coordinate can be simplified to}$$

$$y_n = \frac{r\sqrt{1-r}}{1+(n-1)\sqrt{1-r}}, \quad n=1,2,3,\dots$$

Obviously,  $r_n \rightarrow 0$  and  $(x_n, y_n) \rightarrow (r, 0)$ , and also  $\frac{r_{n+1}}{r_n} \rightarrow 1$  as  $n \rightarrow \infty$ .

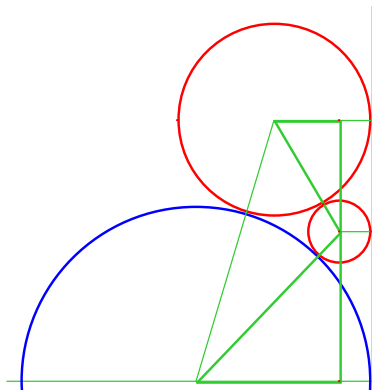


Figure 6. Triangles used for establishing the recursion formula.



By symmetry, the corresponding formulæ for the tangent circle chain to the right Archimedean circle, with radii  $r'_n$ , are

$$r'_{n+1} = \frac{(1-r) \cdot r'_n}{(\sqrt{1-r} + \sqrt{2r'_n})^2}, \quad n=1,2,3,\dots, \text{ and } r'_n = \frac{r(1-r)}{2(1+(n-1)\sqrt{r})^2}, \quad n=1,2,3,\dots,$$

$$\text{and the centre coordinates } \begin{cases} x'_n = r + r'_n \\ y'_n = \sqrt{2(1-r) \cdot r'_n} \end{cases}, n=1,2,3,\dots,$$

where the y-coordinate can be simplified to  $y'_n = \frac{(1-r)\sqrt{r}}{1+(n-1)\sqrt{r}}$ ,  $n=1,2,3,\dots$

The circle  $C_{C,G}$  is double magic because it can be used to construct both the twin circle and the chain of tangent circles shown in figure 7.

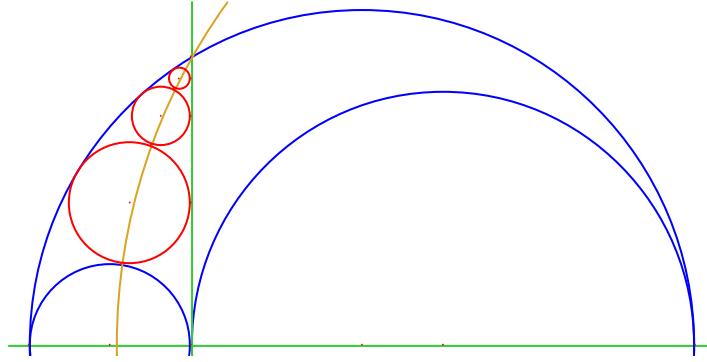


Figure 7. A chain of tangent circles in the *arbelos*.

In figure 7 the radius of the left inner circle of the *arbelos* has been made small to make it possible to visualise the chain of tangent circles having their points of tangency on the circle  $C_{C,G}$  (see notation in figure 3). To prove this tangency property I note that the tangent  $t$  to circle  $C_{C,G}$  is the angle bisector to the line FG and the tangent  $t'$  to circle  $C_{A,C}$  (see figure 8, where P is a point on  $t'$  left to G). Using the notations

$$\begin{cases} \alpha = \angle BGF \\ \beta = \angle BGP \\ \varphi = \angle AGF \end{cases}$$

an algebraic calculation shows that  $\tan \alpha = \tan \beta$ .

Indeed,  $\tan \alpha = \frac{r}{\sqrt{r(1-r)}}$ ,  $\tan \varphi = \frac{r - \frac{1}{2}}{\sqrt{r(1-r)}}$ , and  $\beta + (\alpha - \varphi) = \frac{\pi}{2}$ . Therefore

$$\tan \beta = \frac{1}{\tan(\alpha - \varphi)} = \frac{1 + \tan \alpha \cdot \tan \varphi}{\tan \alpha - \tan \varphi} = \frac{r}{\sqrt{r(1-r)}}, \text{ after simplification.}$$

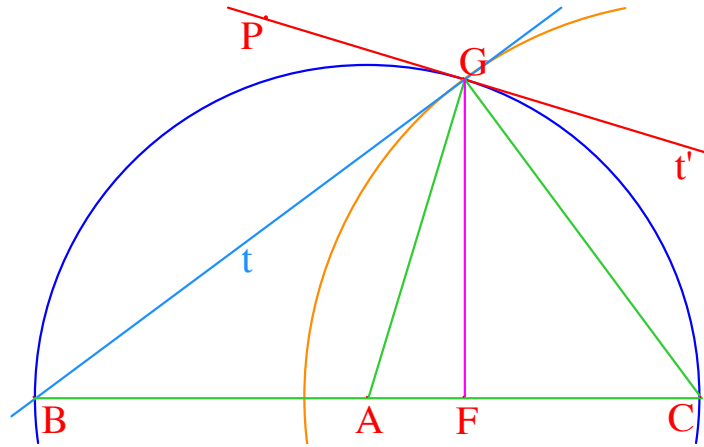


Figure 8. Angle bisector.

The tangency property to be proved now follows by circle inversion of the line by FG and the circles  $C_{A,C}$  and  $C_{C,G}$  in the circle  $C_{G,F}$ . Since the point G by this inversion is mapped to infinity the line by FG and the circles  $C_{A,C}$  and  $C_{C,G}$  all are mapped on lines, intersecting at the mid point on FG with the image of the circle  $C_{C,G}$  being the bisector to the other two lines. This proves the tangency property, illustrated in figure 9 (where  $r$  is small to make it possible to visualise the inversion), where Archimedes' circle and one more circle in the chain and its inverses are shown. The left inner circle of the *arbelos* is mapped onto itself, and the line by the diameter BC onto the circle with diameter FG.

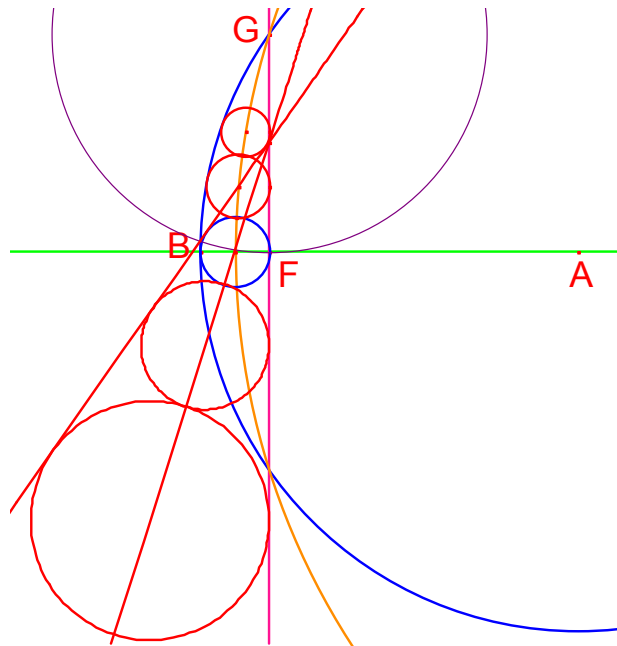


Figure 9. Inversion of circle chain.

By elementary geometry, using algebra, a recursion formula for the radii of the circle chain in figure 7 can be established. Setting the Archimedean circle as the first circle in this chain, the result of this computation is the formula

$$r_{n+1} = \frac{r_n(1-r)}{(1+\sqrt{r-2r_n})^2}, \text{ for } n=1,2,3,\dots \text{ with } r_1 = \frac{r}{2}(1-r).$$

By this recursion formula the following explicit formula can be proved by mathematical induction<sup>8</sup>:

$r_n = \frac{r}{2} \cdot \frac{(1-r)^n}{P_n^2(r)}$ ,  $n=1,2,3,\dots$ , where  $P_n(r) = \frac{1}{2} \left( (1+\sqrt{r})^n + (1-\sqrt{r})^n \right)$ , which is a polynomial in  $r$  of degree  $[n/2]$ . For example, for  $n=2$  and  $n=5$  these radii are

$$r_2 = \frac{r}{2} \cdot \frac{(1-r)^2}{(1+r)^2} \quad \text{and} \quad r_5 = \frac{r}{2} \cdot \frac{(1-r)^5}{(1+10r+5r^2)^2}.$$

To complete the proof by induction the equality  $\frac{r_n(1-r)}{(1+\sqrt{r-2r_n})^2} = \frac{r}{2} \cdot \frac{(1-r)^{n+1}}{P_{n+1}^2(r)}$

must be established with  $r_n = \frac{r}{2} \cdot \frac{(1-r)^n}{P_n^2(r)}$ , which means that the equality

$\sqrt{rP_n^2(r) - r(1-r)^2} = P_{n+1}(r) - P_n(r)$  must hold. This is easily seen, since the right hand side equals  $\frac{\sqrt{r}}{2} \left( (1+\sqrt{r})^n - (1-\sqrt{r})^n \right)$ , and

$$P_n^2(r) - (1-r)^2 = \frac{1}{4} \left( (1+\sqrt{r})^{2n} + (1-\sqrt{r})^{2n} + 2(1-r)^n \right) - (1-r)^n = \left( \frac{(1+\sqrt{r})^{2n} - (1-\sqrt{r})^{2n}}{2} \right)^2$$

It is obvious that for  $r > \frac{1}{2}$  these circles are very small. For example,  $r = \frac{2}{3}$  gives  $r_2 = \frac{1}{75}$

and already  $r_3 = \frac{1}{729}$ , as compared to the diameter  $BC = 1$ . From the recursion formula it

follows directly by mathematical induction that  $r_n < \frac{r}{2} \cdot (1-r)^n$ , and therefore that for

---

<sup>8</sup> I'm grateful to Thomas Bäckdahl for finding the explicit formula for the polynomial  $P_n$  from the number pattern of the coefficients of the first ten polynomials.

large  $n$  the quotient  $\frac{r_{n+1}}{r_n} < \frac{(1-r)}{\left(1 + \frac{1}{2}\sqrt{r}\right)^2}$ , which means that the radius is decreasing more

rapidly than by a geometric progression.

The corresponding circle centres  $(x_n, y_n)$  can be expressed by  $r$  and  $n$  by using the expression for  $r_n$  above and the geometrically derived relations

$$\begin{cases} x_n = r - r_n \\ y_n^2 = (1 - r)(r - 2r_n) \end{cases}$$

It is obvious that these points indeed approach the point G (see figure 9) as  $n \rightarrow \infty$ .

### Magic circle 3

The circles  $C_{B,F}$  and  $C_{C,G}$  meet in R. Draw the perpendicular bisector to BR, meeting BC in H. Draw the circle  $C_{H,R}$ . This circle is orthogonal to any circle tangent to both circles  $C_{A,B}$  and  $C_{D,B}$ , and I therefore call it a magic circle. To prove this orthogonality property, use circle inversion in the circle  $C_{B,F}$  (see figure 10).

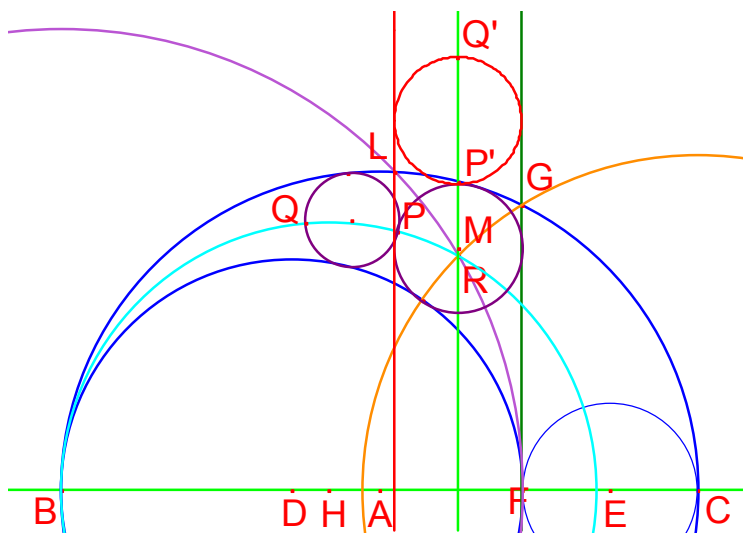


Figure 10. Inversion in circle  $C_{B,F}$ .

Since the point B is mapped to the infinity point by this inversion, the circle  $C_{D,B}$  is mapped to the line by FG (and vice versa), and the circle  $C_{A,B}$  is mapped to the line perpendicular to BC passing L (the point of intersection between the circles  $C_{A,B}$  and

$C_{B,F}$ ), and the circle  $C_{H,R}$  to the line perpendicular to BC passing the point R of intersection between the circles  $C_{H,R}$  and  $C_{B,F}$ . The  $x$ -coordinate of R is easily calculated to  $\frac{r}{2}(1+r)$ , and of L to  $r^2$ , showing that Archimedes' circle is tangent to the line by L perpendicular to BC, having its centre on the line by R perpendicular to BC. This also proves the assertion above, illustrated in figure 10 by the points P and Q and their inverted points P' and Q'. The twin circle is mapped onto itself.

The radius of the magic circle  $C_{H,R}$  is by elementary algebra found to be  $\frac{r}{1+r}$ , which is the harmonic mean of the radii of the circles  $C_{D,F}$  and  $C_{A,C}$ . As known, the harmonic mean is also connected to Archimedes' circles: the diameter of the Archimedean circle is the harmonic mean of the radii of the circles  $C_{D,F}$  and  $C_{E,F}$ .

I also note that the magic circle  $C_{H,R}$  meets the circle  $C_{E,C}$  at a tangency point of the incircle of the *arbelos*, and so does the corresponding magic circle with a radius that equals the harmonic mean of the circles  $C_{E,C}$  and  $C_{A,C}$  meet the other tangency point of the incircle (see figure 11). This is a direct consequence of the orthogonality property of these magic circles I discussed above, and provides one way of constructing this incircle<sup>9</sup>. The circle  $C_{H,R}$  is thus also double magic. It can also be observed that the line passing the two intersection points of the magic circles  $C_{H,R} = C_{H,B}$  and  $C_{H',C}$  passes the centre of the *arbelos* incircle. Using the radius  $HB = \frac{r}{1+r}$  and the radius  $H'C = \frac{1-r}{2-r}$ , the  $x$ -coordinate of these intersection points can be algebraically determined to be  $\frac{r(1+r)}{2(1-r+r^2)}$ , which equals the  $x$ -coordinate of the centre of the incircle (see [1]).

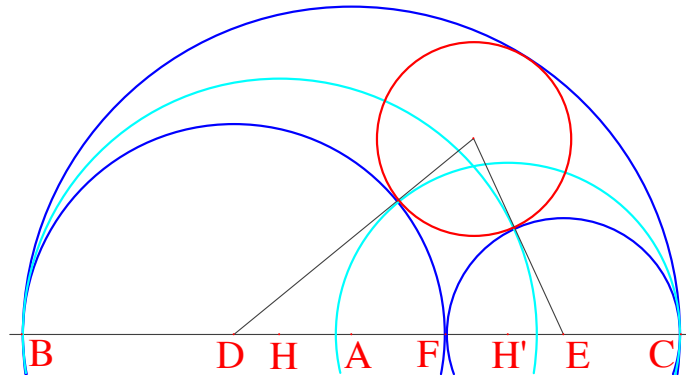


Figure 11. Magic circles and the *arbelos* incircle.

<sup>9</sup> See for example [4] for some simple constructions of the *arbelos* incircle.

An algebraic calculation, using Pythagoras' theorem and the length of the radius HB, shows that the centres  $(x_n, y_n)$ ,  $n=1,2,3,\dots$  of the circles in the circle chain orthogonal to  $C_{H,R}$  and tangent to the circles  $C_{D,F}$  and  $C_{A,C}$  can be expressed by their radii  $r_n$  with the formulae

$$\begin{cases} x_n = r_n \cdot \frac{1+r}{1-r} \\ y_n^2 = r \cdot \frac{2r_n}{1-r} \cdot \left(1 - \frac{2r_n}{1-r}\right) \end{cases}$$

where the circle for  $n=1$  may be, for example, the incircle of the *arbelos*, or the left Archimedean circle<sup>10</sup>. With this latter choice of  $r_1 = \frac{r}{2}(1-r)$  an algebraic computation

yields that  $r_2 = \frac{(1-r)(2-2r+r^2 \pm 2(1-r)\sqrt{1-r})}{2(4-4r+r^3)}$ , where the sign is related to which side

of the Archimedean circle the tangent circle is located. The general explicit formula for the radius  $r_n$  seems to be complex<sup>11</sup>. It can also be observed that for  $r = \frac{3}{4}$ , the right-sided

tangent circle to Archimedes' circle is the *arbelos* incircle, then having the radius  $\frac{3}{26}$ , and the magic circle  $C_{C,G}$  then having the same radius as the basic circle  $C_{A,C}$  (see figure 12).

---

<sup>10</sup> It is easily checked that these circle centres  $(x_n, y_n)$ , for any choice of  $r_n$ , lie on the ellipse

$$\frac{\left(x - \frac{1+r}{4}\right)^2}{\left(\frac{1+r}{4}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{r}}{2}\right)^2} = 1 \text{ (most often discussed in relation to } Pappus \text{ chain; see [1])}$$

<sup>11</sup> In the case with *Pappus chain*, where  $r_1$  is the radius of the *arbelos* incircle, the formula is

$$\text{known to be } r_n = \frac{r(1-r)}{2(r + n^2(1-r)^2)}. \text{ See [1].}$$

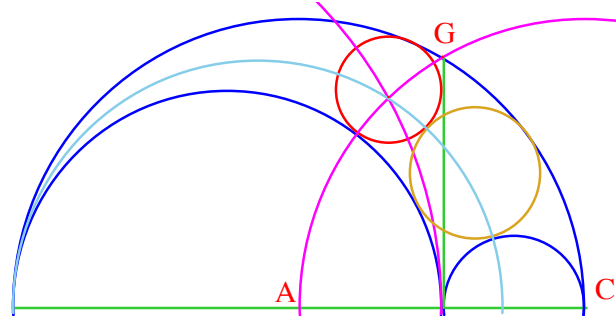


Figure 12. Magic circles with arbelos incircle tangent to Archimedes' circle.

For the construction of this circle chain, draw the line by R and Q (as in figure 10), and draw the line by A parallel to RQ, meeting the circle  $C_{A,C}$  in P (see figure 13). The line by P and Q meets the circle  $C_{A,C}$  in P', which then must be a tangency point for the next circle in the chain. Therefore, the intersection point of the radius AP' and RQ produced will meet in the centre point N of the tangent circle, which thus can be drawn.

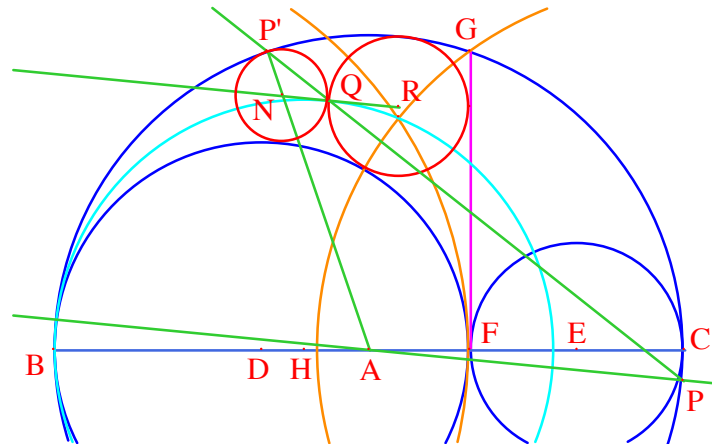


Figure 13. Construction of circle chain.

Another observation points to the fact that the line by R and R', meeting FG at U, is perpendicular to the segment AU (see figure 14), since R and R' are equidistant from FG (as shown above) and the segments AR and AR' both are equal to  $\frac{1}{2}\sqrt{r^2 + (1-r)^2}$ . This means that the segment QQ' is divided into two equal parts by the segment FG.

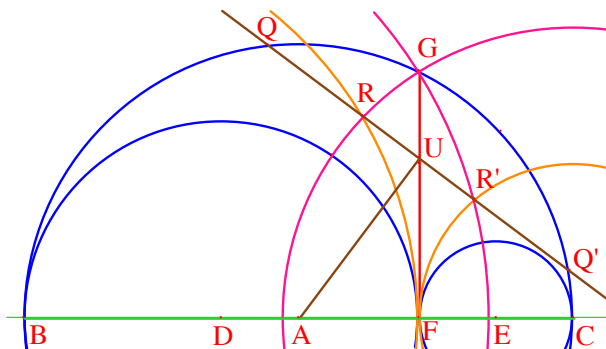


Figure 14. Segments  $RR'$  and  $AU$  are perpendicular.

I also note the circle  $C_{A,R}$  passes the midpoints of the circle arcs  $BF$  and  $FC$ , respectively, and has the same area as the circle passing the midpoint of the arc  $BC$ , and the points  $B$  and  $F$  (see figure 15). These observations are established by simple algebraic calculations.

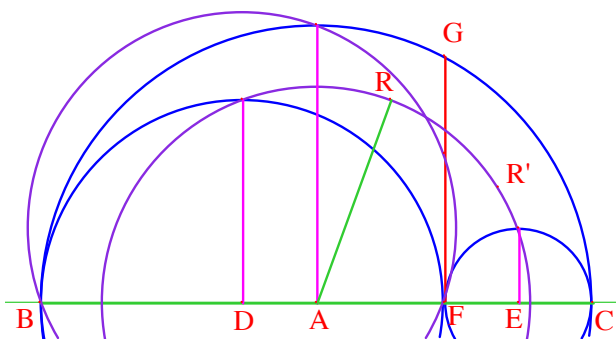


Figure 15. Two equal circles by arc midpoints.

A simple algebraic calculation also shows that the circle ring between the circles  $C_{B,G}$  and  $C_{B,F}$  has the same area as the circle ring between the circles  $C_{C,G}$  and  $C_{C,F}$  (see figure 13), equal to  $\pi r(1 - r)$ , which equals the area of the circle with radius  $FG$ .

### The magic circles and circle chains – a summary

In figure 16 the three magic circles and the circle chains related to them are shown. Their constructions and some of their characteristics, including some algebraic representations,



have been presented above. Similarly, the corresponding circles can be constructed on the right side of the perpendicular segment in the *arbelos*. Archimedes' circle thus serves as the starting circle of three different circle chains in the *arbelos*, associated with each of the three magic circles. It has been shown above that the intersection point of these three circles and the centre of Archimedes' circle lie on the same perpendicular line to the common diameter of the *arbelos* circles. It has also been observed that that two of the magic circles meet the inner circles of the *arbelos* at the tangency points of the *arbelos* incircle. Some additional observations have also been presented.

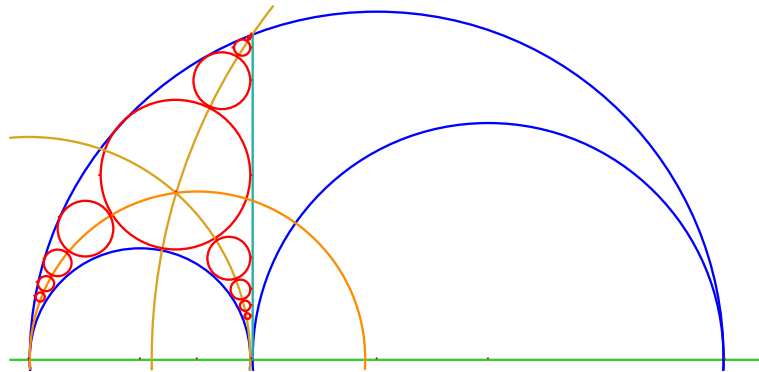


Figure 16. Magic circles and circle chains.

## References

- [1] van Lamoen, F. and Weisstein, E. W. : "Pappus Chain." From *MathWorld* - A Wolfram Web Resource.  
[Retrieved on Dec. 7, 2006, at <http://mathworld.wolfram.com/PappusChain.html> ]
- [2] Schoch, T.: "Arbelos: The Woo circles".  
[ Retrived on Dec. 7, 2006, at <http://www.retas.de/thomas/arbelos/woo.html> ]
- [3] Weisstein, E. W.: "Arbelos." From *MathWorld* - A Wolfram Web Resource.  
[Retrieved on Dec 7, 2006, at <http://mathworld.wolfram.com/Arbelos.html> ]
- [4] Woo, P.: "Simple Construction for the Incircle of an Arbelos." *Forum Geometricorum* **1** (2001), 133-136.
- [5] Bergsten, C.: Magic circles in the arbelos. *LiTH-MAT-R-2006-12*, Department of Mathematics, Linköpings universitet.
- [6] Danneels, E. & van Lamoen, F. : Midcircles and the Arbelos. *Forum Geometricorum* **7** (2007), 53-65.

## Chaos in Physics and Recurrence in Arithmetic Sets

Jean Dayantis<sup>1</sup>  
Pignan, France

**Abstract:** After briefly recalling the concepts of recurrence and chaos in physics, the recurrence properties of arithmetic sets are examined following Gauss' method, as exposed in part three of his *Disquisitiones Arithmeticae*. This problem in number theory is related to the physical problem of recurrence in deterministic chaos. Most possible forms of moduli are examined in detail with respect to their recurrence properties, for application to the generalized Bernoulli mapping. The emphasis is put on period lengths, rather than on congruences. In an annex the recurrence properties of Arnold's cat map are briefly examined.

Keywords: Arnold's cat map; *Disquisitiones Arithmeticae*; deterministic chaos; number theory; period lengths; recurrence

### INTRODUCTION

#### A- The concept of recurrence

Recurrence is a very general concept and pervades almost every field of science : Astronomy, Physics, Chemistry, Geology, Biology and so on. In everyday language, recurrence means that something that has happened in the past will again happen in the future, and this many many times, if not an infinity number of times. Let us first give some examples: in astronomy, the revolution of the earth around the sun, responsible for the succession of seasons, is a recurring phenomenon; in the same way, the days in the week or the months in the year are recurrent; in physics, any oscillatory process, for example the frictionless pendulum is a recurrent process; in thermodynamics, the Carnot cycle is recurrent phenomenon, in so far as the engine is given the necessary feed to sustain its motion. In chemistry, the Belousov-Zhabotinski<sup>1</sup> reaction in far from equilibrium thermodynamics, is, among other chemical reactions, a recurrent one. In medicine and biology, the cardiac rhythm, the rhythm of breathing are recurrences. In dynamics, there is a theorem, the Poincaré recurrence theorem (1890), which states that "in a dynamical system having constant energy, any point of the trajectory in phase space<sup>2</sup> will again be approached as closely as wanted with time".<sup>3</sup> The fact that this Poincaré recurrence time may, in some instances and for all practical purposes, be infinite, shall not concern us here.

Of course, many if not most phenomena in nature are not recurrent. They are then said to be irreversible. For example, in chemistry, an explosion is not a recurrent phenomenon, since the initial products are consumed during the explosion to give final products which are stable in time; in the same way, the evolution of most chemical reactions is monotonous, heading to equilibrium; in astronomy, the life and death of a star is not recurrent, the stars follow roughly speaking the so-called Herzprung-Russel diagram (1911-1913)<sup>4</sup>, being blue

---

<sup>1</sup> \* On leave from Institut Charles Sadron, Campus CNRS de Cronenbourg, 13, rue du Loess, 67034 Strasbourg Cedex France. E-mail : [jeandayantis@aol.com](mailto:jeandayantis@aol.com). Any correspondence should be addressed to the following address 12, rue de l'Olivette, 34570 Pignan France

stars when born through accretion of matter and becoming white dwarfs, neutron stars or black holes with their death, according to the magnitude of their masses. The same may be said in biology with regard to the living organisms on earth, which are born, prosper and finally die.

Are there any recurrences in mathematics? Of course there are. In this article however the subject of recurrences in mathematics will be restricted to some properties of integer numbers, and this in relation to chaos theory.

## B- Chaos theory.

It is outside the scope and the purpose of the present paper to give even a limited account of chaos theory. However, as the inception of this paper is related to chaos in physics, a few general notions will be here recalled.<sup>5</sup>

The intuitive, everyday language concept for “chaos” refers to a system where no order is apparent, one which appears to be very “disordered”, one which does not seem to follow any law whatsoever. A more scientific definition is that, if a particular element in the system is chosen, the autocorrelation function<sup>6</sup> of this element as a function of time is of finite amplitude and tends more or less rapidly to zero. For example, choose a particular molecule in a gas enclosed in a container, large with respect to the dimensions of the molecule, and let  $x_0$ ,  $y_0$ ,  $z_0$  be its coordinates at time  $t_0$ . It is assumed that the container plus the enclosed gas molecules form an isolated thermodynamic system, i.e. no exchange of matter or energy with the environment does occur. If at time  $\tau_1$  the coordinates are  $x_1$ ,  $y_1$ ,  $z_1$ , and if one is unable to determine these coordinates from the previous  $x_0$ ,  $y_0$ ,  $z_0$ , the autocorrelation function<sup>5</sup> is said to have reached the value zero at time  $\tau_1$ . In other words, the “memory” of the system is lost, and the past does not define the future. This is a concept of particular importance.

Until near the end of the second third of the twentieth century scientists thought that chaos in physical systems resulted from the existence of a very large number of degrees of freedom. Thus, let the above container embody  $N$  molecules of a gas so that the system has  $6N$  degrees of freedom. If  $N$  is equal to Avogadro’s number, then  $N=10^{23}$  molecules. If the position and velocity of each one of these molecules could be exactly known at a time say  $\tau_0$ , then, according to Newtonian mechanics and at least in principle, the position and velocity of each of these  $N$  molecules could be calculated at a later time  $\tau_1$ .<sup>7</sup> In other words, the future evolution of the system would be entirely determined by its past history. However, as in practice it is not possible to solve at time  $\tau_0$  some  $10^{23}$  scalar equations to find the positions and velocities of the  $N$  gas molecules at time  $\tau_1$ , one is happy to be able to specify only values for some global parameters in the gas, as the mean temperature or the mean pressure. In other words, the “chaos” in the positions and velocities of the gas molecules at any time results here from a practical rather than a theoretical impossibility: the impossible knowledge of the trajectory of the physical system in phase space is here due to the large number of degrees of freedom involved. However, this is not always so, and there are many physical systems where the unpredictability of the evolution is of an intrinsic nature.

Such physical systems do not require a large number of degrees of freedom. Three are sufficient. These physical systems are mathematically described by systems of non-linear differential equations which are generally non-integrable, that is they have no analytical solution. The solutions can be found only numerically, using a step by step process in the variables, a procedure which generally necessitates the use of computers. This explains why such numerical solutions had to wait for the advent of computer science and computers.

The first physical system having only a few degrees of freedom and yet displaying chaos was discovered in 1963 by the American meteorologist Edward Lorenz.<sup>8</sup> The simplified model he used to follow the evolution of weather had only three degrees of freedom, yet the system was chaotic, it did not follow any regular law, which means that the weather is by essence unpredictable for long enough periods of time. Since Lorenz's discovery, many works have followed concerning physical systems having only a small number of degrees of freedom and being nevertheless chaotic. The chaos thus generated by systems displaying only a limited number of degrees of freedom is referred to as "deterministic chaos".<sup>9</sup> All such deterministic chaotic systems are "dissipative", which means that they "dissipate" energy: they are not thermodynamically isolated, they exchange matter or energy or both with the environment, contrary to the previous example of a gas enclosed inside a container.

A constant feature of chaotic systems is their instability, that is their sensitivity to initial conditions: if we very slightly vary the initial conditions, in other words the departure point in phase space, then the subsequent trajectories will wildly differ. As was aptly stated by Lorenz, "The flapping of a single butterfly's wing today produces a tiny change in the state of the atmosphere. Over a period of time, what the atmosphere actually does diverges from what it would have done. So, in a month's time, a tornado that would have devastated the Indonesian coast doesn't happen. Or maybe one that wasn't going to happen, does."<sup>10</sup>

#### C- Deterministic chaos and mapping.

Because of the difficulty of studying actual physical systems with respect to chaos and possible recurrence to the initial state (e.g. in Hamiltonian Mechanics the Poincaré theorem, see above), physicists and mathematicians have sometimes resorted to artificial models of chaos, in order to capture at least some features of the true, physical chaos. Such models are constructed using "mapping", that is an application of a given (phase) space to itself. The simplest possible example of such mapping is the Bernoulli mapping:<sup>11</sup> consider the linear segment  $[0,1]$ , (here, the phase space) and take some point  $x_N$  inside this segment. The mapping consists in taking as next point  $x_{N+1}=2x_N$  if  $2x_N < 1$ , and  $x_{N+1}=2x_N-1$  if  $2x_N > 1$ . It is demonstrated that after a sufficient number of steps, the linear segment  $[0,1]$  will be uniformly covered by the step points. This Bernoulli mapping is not reversible, since if one takes  $x'=x/2$ , eventually, whatever the starting point  $x$ , the mapping will lead to the origin  $x=0$ . Nevertheless, it is recurrent (see part A.1 of the main text). Other "popular" mappings are the baker's map and Arnold's cat map<sup>12</sup> (See Annexe III). The present article however is particularly devoted to the case of the Bernoulli mapping.

### RECURRENCE IN ARITHMETIC SETS

After this somewhat lengthy but necessary introduction in order to place the article in its context, it is time to proceed to the gist of the matter. This is the recurrence in arithmetic sets, and the topic is closely related to congruences in the theory of numbers.

The study of the recurrence in numerical sets involves considering the  $\mathbb{Z}/\mathbb{Z}(N)$  algebra<sup>13</sup>. Leonhard Euler (1707-1783) was the first to consider, to our knowledge, congruences modulo some integer and also recurrences, and he introduced the notion of primitive roots (see below). However the in depth study of recurrences and primitive roots is due to Carl Friedrich Gauss (1777-1855) in his *Disquisitiones Arithmeticae*<sup>14</sup> (denoted below D.A.). He was followed by many celebrated mathematicians as Poincaré, Jacobi and Tchebycheff and numerous other distinguished mathematicians.<sup>15</sup> However, the focus of these authors was essentially the finding of primitive roots, not the length of sequences. The latter was not considered to be of particular interest.

As already started in the introduction, the problem here considered is to analyse the status of recurrence in deterministic mappings. To give a striking example, let us consider Arnold's cat map<sup>12</sup>: In this particular one to one transformation, after a relatively small number of successive transformations, the original picture of the cat completely disappears and chaos is being established. However, pursuing further the transformations, the picture of the cat finally reappears identically. This may take as many as several hundred of successive transformations, or even more. Though the reappearance of the cat's picture may at first seem something miraculous, the mathematical explanation of this remarkable phenomenon is quite simple. (See Appendix III).

Many of the results to follow are already present in the D.A., published more than two centuries ago. The derivations however are sometimes somewhat different, and it is hoped that this may provide some new insights. The elementary axiomatic status of this work will hopefully be easy and useful reading to physicists, non-career or amateur mathematicians, as also all propositions are clarified by numerical examples. However, some questions remain open and it is hoped that this might attract the interest of mathematicians for further clarification. (See e.g. the conjecture in C.1.2 or the last sentence in Part D). Because of the purpose of the paper, the emphasis is put on recurrences and length of periods, rather than on congruences to unity or other integers, as is the case in many classical works. Gauss' terminology will be followed throughout.

## PART A. RECURRENCE MODULO A PRIME P.

Consider an odd prime number  $p$ , an integer  $\alpha < p$  not divisible by  $p$ , and the successive powers  $\alpha^1, \alpha^2 \dots \alpha^{p-1}$  of  $\alpha$ , designed as the base. Euler and Gauss have shown<sup>16</sup> that quite generally, for the power  $t$  of  $\alpha$ , called the index,  $t$  residues taken among the  $p-1$  possible appear modulo  $p$  (mod. $p$  in what follows) before the series repeats itself. The number  $t$  is necessarily a divisor of  $p-1$ . If the repetition occurs at the power  $t = p-1$ , then all integers from 1 to  $p-1$  appear in the period as residues (mod. $p$ ), and  $\alpha$  is then called a *primitive root* of  $p$  (in what follows PR). For example, one easily finds that 2 is a primitive root of five, but not of seven. It has been demonstrated by Euler and Gauss that any odd prime has primitive roots. In what follows, only the least residues (mod. $p$ ) are considered.

### A.1. The base $\alpha$ is a primitive root of $p$ , an odd prime.

According to convenience, enumeration of the elements in the set may start either from  $\alpha^1$  (enum.1), or from  $\alpha^0=1$  (enum.2). Using here enum.1, one has for the last term in the series before repetition, from Fermat's theorem,  $\alpha^{p-1} \equiv 1 \pmod{p}$ . This involves that one necessarily has for the term  $\alpha^{(p-1)/2} \equiv \pm 1 \pmod{p}$ , for this is the only way by squaring each member of the above relationship to obtain  $\alpha^{p-1} \equiv 1 \pmod{p}$ . The value  $+1 \pmod{p} = \alpha^0$  is excluded, since then the series would stop and recurrence will occur at the term  $\alpha^{(p+1)/2}$ , and therefore  $\alpha$  will not be a primitive root, contrary to hypothesis. Consequently, one necessarily has  $\alpha^{(p-1)/2} \equiv -1 \pmod{p}$ <sup>17</sup> and this says that the residue of the term  $\alpha^{(p-1)/2} \pmod{p}$  is equal to  $p-1$ . This is easily checked. (Take for example  $\alpha=2$  and  $p=13$ , or  $\alpha=3$  and  $p=17$  and write down the period). Let us then write the set of integers in the period (mod. $p$ ) as follows :

$$\alpha^1, \alpha^2, \dots, \alpha^{(p-3)/2}, \overset{\equiv p-1}{\alpha^{(p-1)/2}}, \alpha^{(p+1)/2}, \alpha^{(p+3)/2}, \dots, \alpha^{(p+(p-4))/2}, \overset{\text{last term} \equiv 1}{\alpha^{(p+(p-2))/2}} \pmod{p} \quad (1)$$

This set can alternatively be written in the form

$$\alpha^1, \alpha^2, \dots, \alpha^{(p-3)/2}, p-1, \alpha(p-1), \alpha^2(p-1), \dots, \alpha^{(p+(p-4))/2}(p-1), 1 \pmod{p} \quad (1')$$

Now it is evident that  $\alpha + (-\alpha) \equiv p \equiv 0 \pmod{p}$ , and more generally

$$\alpha^v + (-\alpha^v) \equiv 0 \pmod{p}. \quad (2)$$

If in the above set we add two by two the terms  $\alpha^v$  ( $v < (p-1)/2$ ) and  $\alpha^{v+(p-1)/2} = \alpha^v(p-1)$ , the result is

$$\sum_{v=1}^{v=(p-1)/2} [\alpha^v p + (\alpha^v + \alpha^{v+(p-1)/2})] \equiv 0 \pmod{p} \quad (3)$$

since the addition of the terms  $\alpha^v$  and  $-\alpha^v$  is, according to (2), equal to  $p^{18}$ . Relationship (3) provides another proof of Gauss' proof,<sup>19</sup> that the algebraic sum of all the terms in the set (1) is zero (mod.p). Gauss' proof is of course the shortest. It is as follows : using enum.2 and writing the successive terms, one has :

$$1 + \alpha^1 + \alpha^2 + \dots + \alpha^{p-1} = (\alpha^p - 1) / (\alpha - 1) = 0 \pmod{p} \text{ since } \alpha^p = 1 \quad (4)$$

Now it is evident, since  $\alpha^v \equiv \alpha^v - mp \pmod{p}$ , where the integer  $m = 0$  if  $\alpha^v < p$ , that replacing the terms in (3) by their value mod.p leads to relationship (4).

Nevertheless, relationship (3) is useful in providing a somewhat better insight on the structure of the period, and the property that the term of index  $(p-1)/2$  equals  $p-1$  will be used later on.

Example : take  $p=11$ , base 2. The residues in the period using enum.1 are, 2,4,8,5,10,9,7,3,6,1, then recurrence occurs. One has, according to (2),  $2+9=11$ ,  $4+7=11$ , etc, the sum of all the terms in the period is  $p(p-1)/2 = 10 \times 11 / 2 = 55 \equiv 0 \pmod{11}$ . Also, the multiplicity  $m$  is easily determined for the term  $\alpha^{(p-1)/2}$  through

$$m = (\alpha^{(p-1)/2} + 1 - p) / p \quad (5)$$

For  $p=11$ ,  $\alpha=2$ , one correctly finds  $m=2$ .

As a practical application, let us consider the simple one dimensional Bernoulli mapping  $X_{N+1} = 2X_N \pmod{1}$ . Starting from the abscissa  $1/11$ , recurrence will occur after 10 steps, the numerator taking all the above mentioned values for  $p=11$ , base  $\alpha=2$ . Now one may of course choose as the denominator an integer for which 2 is not a primitive root, e.g. 17, and initiate the process at abscissa  $1/17$ . In this case recurrence will not occur after  $p-1$  steps, but only  $(p-1)/2$ . Also the denominator may be taken a composite integer, for example  $3 \times 11 = 33$ ,  $3 \times 5 \times 7 = 105$  or  $2^2 \times 13 = 52$ , an even number. Therefore it is interesting to know the recurrence properties in these cases also, which will be examined in the following Parts and sections, excluding however irrational denominators.

## A.2. The base $\alpha$ is not a primitive root of $p$ , an odd prime. Open and closed groups.

If  $\alpha$  is not a PR of  $p$ , then the period stops and recurrence is initiated, as shown by Gauss,<sup>20</sup> for an index  $v$  which divides  $p-1$ . If  $p-1$  is of the form  $p-1=2p'$ ,  $p'>3$  prime, then the period has necessarily  $p' = (p-1)/2$  terms. This is a sufficient but not a necessary condition in

order that the period be of  $(p-1)/2$  terms. The alternative of having  $p'$  groups of two terms beginning with  $\alpha^0 \equiv +1$  and  $\alpha^1 \equiv p+1$  is impossible since necessarily  $a < p$ . Now since the period has only  $(p-1)/2$  residues and there are  $p-1$  integers from 1 to  $p-1$ , there are necessarily  $(p-1)/2$  integers lacking in the period. The group of integers constituted by the  $(p-1)/2$  residues corresponding to the actual powers taken by  $\alpha$  before recurrence, will be called the *principal group*, noted Gr1.

This group includes necessarily unity. If further  $p-1=2p'$ ,  $p'$  prime, there can be only another group which shall be called the *secondary group* Gr2, containing the  $(p-1)/2$  integers absent in Gr1. In other situations, however, there may be more than one secondary group (see below, paragraph A.3). How one will find the elements of Gr2 ? Choose the least prime  $\omega$  not included in the period of Gr1, and multiply  $\omega$  by  $\alpha, \alpha^2, \dots, \alpha^{(p-1)/2}$ . One obtains in this way  $(p-1)/2$  new integers (mod.p) all necessarily different from those of Gr1, which will be called "residues" to distinguish them from the residues of Gr1. For, suppose that Gr2 contains a term  $\beta$  already appearing in Gr1. Then necessarily one has  $\beta = \omega \alpha^v = \alpha^\mu$  with  $\mu > v$ , since  $\omega > 1$ . By dividing both members of this relationship by  $\alpha^v$  one has  $\omega = \alpha^{\mu-v}$ , which means that  $\omega$  is a residue of Gr1, contrary to hypothesis. Since there are in all  $p-1$  integers in the range  $1 \dots p-1$  (mod.p), the sum of the integers in Gr1 and Gr2 completes the set of all possible integers from 1 to  $p-1$ . Though it is convenient to choose  $\omega$  as the least prime among those not included in Gr1, this is not a necessary condition. On the contrary,  $\omega$  should never be chosen to be a composite integer not in Gr1, since this will lead to redundancies. Another basic difference between the principal group Gr1 and the secondary group Gr2 (beyond the fact that the terms in this group are not residues (mod.p) of the powers of  $\alpha$ ), is that the elements of Gr2 in non modular algebra have always in factor  $\omega$ , which is never the case for those in Gr1. Notice that the notation as groups for GrX complies to the usual definition of groups in mathematics.

Examples will make the above clear: Take  $p=23$ , base 2, so that  $p-1=2 \times 11$ , and  $p'=11$ . One then finds in the period of Gr1 the residues  $[2,4,8,16,9,18,13,3,6,12,1]$ . To find the "residues" of Gr2, choose 5, the least prime not included in the period of Gr1, and multiply 5 by  $2, 2^2 \dots 2^{(p-1)/2}$ . This leads to the period  $[5,10,20,17,11,22,21,19,15,7,17]$ . Examination of these two periods shows that the residue  $p-1$  appears in the group Gr2 and that all the elements of Gr2 are obtained by subtracting from 23 the elements of Gr1. More generally, within the constrain indicated, the terms in Gr2 are obtained by subtracting from  $p$  the residues of Gr1. If  $p - \alpha^v$  is such a term of Gr2, it cannot be also a residue  $\alpha^m$  of Gr1, which will lead to  $\alpha^v + \alpha^m = p$ . In the present case, no addition of terms appearing only in either Gr1 or Gr2 may add up to  $p$ .

PROOF : Suppose that two residues of Gr1,  $x_1$  and  $x_2$ , were such that  $x_1 + x_2 = p$ . The corresponding terms in Gr2 are then  $y_1 = p - x_1$  and  $y_2 = p - x_2$ , which added lead to  $2p = x_1 + x_2$ , a relationship not compatible with the previous one. Only one term of Gr2 when added to the corresponding residue of Gr1 makes up to  $p$ .

Such couples of groups Gr1 and Gr2 will be called *open*, since there is interconnection of the elements of each group with those of the other. As already stated, in *open* systems, no sum of two terms in the same group adds up to  $p$ .

Consider now  $p=17$ ,  $p-1=16=2^4$  and the base  $\alpha=2$ . This base is not a PR of 17, as  $2^8 = 256 = 2^{(p-1)/2} \equiv +1 \pmod{17}$ . In principle  $p-1$  might decompose to a principal group Gr1 of eight, or four elements, but the latter is impossible since  $2^4 = 16 < 17$ . Writing down the period leads to a series of eight residues  $[2,4,8,16,15,13,9,1]$  so there must be a group Gr2 having also eight terms. To find these terms take the least prime, here 3, not appearing in the period of the principal group, and multiply it by  $2, 2^2, \dots \pmod{17}$ . One thus obtains the period  $[3,6,12,7,14,11,5,10]$ .

In Gr1, because 4 divides 16 and  $2^{(p-1)/2} \equiv +1$  one necessarily has  $2^{(p-1)/4} \equiv \pm 1 \pmod{17}$ , and since the period continues to eight terms as found above, one has  $2^{(17-1)/4} \equiv -1$ , that is, to follow Gauss' terminology, the residue  $(p-1)/2$  "pertains" to the index 4. Here eq. (2) applies, and the sum of the corresponding terms two by two in Gr1 as well as in Gr2 equals  $17=p$ . Also, one does not here obtain the elements of Gr2 by subtracting to the modulus  $p$  the residues in Gr1. If this is done, one obtains again the residues of Gr1, and vice-versa for the terms in Gr2 ; in other words, the groups are cyclic. Such groups will be called *closed*.

Generally, from the above, the rationality of *open* and *closed* groups is the following : if the period has an even number of terms,  $(p-1)/2$ ,  $(p-1)/4$ , etc. then the term obtained by dividing by 2, of index  $(p-1)/4$ ,  $(p-1)/8$ , etc. has value  $\equiv -1 \pmod{p}$ , or, to use Gauss' terminology, the residue  $p-1$  pertains to the indices  $(p-1)/4$ ,  $(p-1)/8$ , etc. In such occurrences, the group Gr1 is closed, since 1 and  $p-1$  add to  $p$ , the property of *closed* groups. If now the period has an odd number of terms, then  $(p-1)/4$ ,  $(p-1)/8$ , etc. do not correspond to any integer index, and the residue  $p-1$  is shifted to Gr2. In mod. $p$  arithmetic this means that no power of the base  $\alpha$ , yields the residue  $p-1$ . Since unity is always to be found in Gr1, addition of the residue unity of Gr1 to the term  $p-1$  of Gr2 adds to  $p$ , the property of *open* groups. If  $p-1 = 2p'$ ,  $p$  and  $p'$  primes,  $\alpha$  not a PR of  $p$ , there are always two and only two groups, the principal Gr1 and the secondary Gr2, which are either both *open* or both *closed*.

There does not seem to exist any rule, which will permit to predict whether the groups mod. $p$ , base  $\alpha$ ,  $\alpha$  not a PR of  $p$ , are *open* or *closed*. One has to write down the periods. However, at least for small values of the modulus  $p$ , it seems that the groups are more often *open* than *closed*.

In secondary groups, beginning with the prime (that is using enum.1), there is also a rank or "index" in the integers of the period. This rank is here called "index", to distinguish it from the index of the integers of Gr1, since these integers are not actual residues of the powers of  $\alpha$ .

### A.3. $p$ an odd prime, the base $\alpha$ not a primitive root of $p$ , more than one secondary group.

What now about the general case where  $p$  is not such that  $p-1 = 2p'$ ,  $p'$  a prime? Then a priori, all the divisors  $\delta$  of  $p-1$  are possible, leading to periods  $(p-1)/\delta$ , with the restriction that are forbidden those divisors of  $p-1$  leading to periods of  $n$  terms for which  $\alpha^n < p$ . Of course, this means that one may have, besides the principal, 2,3 ... secondary groups, in which groups, together with the principal, all the integers from 1 to  $p-1$  will be present. To find all the secondary groups, it suffices to extend the procedure indicated in A.2 : Take the least prime  $\omega$  not part of the  $t$  residues in Gr1, and multiply it by  $\alpha$ ,  $\alpha^2$ , ...  $\alpha^t$ , to find the "residues" in Gr2. Then choose the least prime  $\omega'$  not present in Gr1 and Gr2, and multiply it by  $\alpha$ ,  $\alpha^2$ , ...  $\alpha^t$ , to find the "residues" in Gr3. Continue in this way until all the secondary groups are found. Proof that not two of the groups may share a same residue, may follow along the lines developed in A.2. Here are a few examples of prime moduli where more than one secondary group exists :

Consider  $p=43$ ,  $p-1 = 2 \times 3 \times 7$ , and the base 2, not a PR; writing down the principal group one finds the period :

$$[2,4,8,16,32,21,42,41,39,35,27,11,22,1], \text{ mod.43}$$

This period has the even number of fourteen residues, and the residue  $p-1$  pertains to the index  $(p-1)/2 \times 3 = 7$ . The group is *closed*, the residue 42 appears at the index  $14/2=7$ , and there are two secondary groups, which for convenience will be denoted here by the prime initiating the



period, that is Gr3 and Gr7. The elements of these two secondary groups are easily found as above indicated. They are:

[3,6,12,24,5,10,20,40,37,31,19,38,33,23] and [7,14,28,13,26,9,18,36,29,15,30,17,34,25].

Consider now  $p=73$ ,  $p-1=72=2^3 \cdot 3^2$ ; neither 2 nor 3 are primitive roots of 73. Let us first write the period base 2. Principal group: [2,4,8,16,32,64,55,37,1]. There is an odd number of residues and the residue 72 does not appear in the principal group, but in the first secondary group Gr71, the period being [71,69,65,57,41,9,18,36,72]. Adding two by two the terms of given “indices” of two associated groups, will yield the value 73. There are six other secondary groups, namely Gr3, Gr5, Gr11, Gr13, Gr53 and Gr61 which are associated two by two, and summing one term of the first to the corresponding term of the second will yield  $73=p$ . The association is as follows:  $Gr5 \Leftrightarrow Gr55$ ,  $Gr11 \Leftrightarrow Gr13$ , and  $Gr3 \Leftrightarrow Gr61$ . In *open groups* the “residue”  $p-1$  may appear in any group other than the principal and may pertain to any “index”.

Let us now take  $p=73$ , base 3. Here there are six groups of twelve terms each, all groups are *closed*, and the residue  $p-1=72$  appears therefore in the principal group at the index  $12/2=6$ . Here we give the period for only the principal group:

[3,9,27,8,24,72,70,64,46,65,49,1].

The five secondary groups are Gr5, Gr7, Gr13, Gr19 and Gr23.

#### A.4 The number of groups in base $\alpha$ modulo the powers of an odd prime $p$ .

Let us first examine what happens modulo  $p^2$ , the base  $\alpha$  being a PR of  $p$ , an odd prime. Firstly, the largest possible period has  $p(p-1)$  terms, since the multiples of  $p$ , which number  $p$ , cannot be residues. Let us now show that if  $\alpha$  is a PR mod. $p$ , it is also a PR mod. $p^2$ : one first remarks that if a residue is  $\equiv +1 \pmod{p^2}$ , it is also necessarily  $\equiv +1 \pmod{p}$ . The period mod. $p$  is  $p-1$ , so that within the period mod. $p^2$  there are  $p$  periods mod. $p$ . As the last term of the last period mod. $p \equiv +1$ , this establishes the fact that if a residue is congruent to unity mod. $p^2$ , it is also congruent to unity mod. $p$ . Now if  $\alpha$  is a PR mod. $p$ , it is also necessarily a PR mod. $p^2$ . For, let us assume that the period mod. $p^2$  were  $p(p-1)/q$ ,  $q$  being some integer dividing  $p-1$ . This implies that the residue at the index  $p(p-1)/q$  would be  $\equiv +1 \pmod{p^2}$ . However, this index will correspond to an index  $(p-1)/q$  inside some period mod. $p$ , which cannot be  $\equiv +1$ , unless  $q = p-1$ . In the latter case, however, the period mod. $p^2$  would be  $p$ , which again is impossible, because this would imply that two adjacent terms mod. $p$ , of indices  $p-1$  and  $p$  would both be equal to  $\equiv +1$ . The conclusion is that if  $\alpha$  is a PR of  $p$ , it is also a PR of  $p^2$ . This property has long been known.<sup>21</sup>

Example. The period of 5 base 3, a PR, is: [3,4,2,1] and that of 25 is [3,9,2,6,18,4,12,11,8,24,22,16,23,19,7,21,13,14,17,1]. One immediately checks that the residue  $p-1 \pmod{p}$  is at the index  $(p-1)/2=2$ , and that the residue  $p^2-1 \pmod{p^2}$  is at the index  $p(p-1)/2=10$ .

The above considerations generalize to any power of the odd prime  $p$ . The period mod. $p^N$  is  $p^{N-1}(p-1)$ , and if  $\alpha$  is a PR of  $p$ , it will also be a PR of  $p^N$ . As a corollary the term of index  $p^{N-1}(p-1)/2 \pmod{p^N}$  is equal to  $p^N-1$ .<sup>22</sup>

The next step is to see what happens when  $\alpha$  is not a PR of  $p$ . In this case the period will be  $(p-1)r \pmod{p}$ ,  $r$  dividing  $p-1$ . As shown above there will then exist mod. $p$   $r$  separate groups, covering as residues all the integers from 1 to  $p-1$ , that is the principal group Gr1 and  $r-1$  secondary groups GrX. Now mod. $p^2$  there will also be  $r$  groups of equal periods  $p(p-1)/r$ . For, if the periods differed from the above value, the sum of all the residues of all the groups mod. $p^2$  would either exceed or be less than the permissible integers, and in both cases this would be irrelevant. The property holds for any integer power of  $p$ .

As already stated all the above considerations apply with no modification in mod.1 algebra, considering fractional abscissas  $\alpha/p$ ,  $\alpha$  an integer less than  $p$ . This is used in particular by physicists in defining deterministic maps of chaos. As one further example, in base 2, and enum.2, let  $p^2=49$ . Since there are two groups mod.7, there are also two groups mod.49 of periods  $(7 \times 6)/2 = 21$  terms each. One indeed finds two open groups of 21 residues each where  $p^2-1$  is found in the secondary group at the “index” 5. Conversion to mod.1 algebra is readily obtained by dividing each residue by 49.

For the physicist's sake, let us stress that the group he chooses is indifferent regarding the length of the period, as all groups have periods of equal length. If for instance he starts his mapping experiment with  $1/p \Rightarrow \text{enum.2}$ , the residues will be those of the principal group, and the length of the period will be  $p-1$ , if the base is a PR of  $p$ . If he starts from  $p'/p$ ,  $p'$  not to be found in the principal group, the period shall remain the same, only the “residues” are different.

## PART B. RECURRENCE IN COMPOSITE MODULI C OF ODD PRIMES.

Let us first quote from Hardy and Wright the following excerpt, regarding the primitive roots of a congruence : *In what follows we suppose that the modulus  $m$  is a prime ; it is only in this case that there is a simple general theory.*<sup>23</sup> Regarding however the recurrence in composite moduli  $C$  which are the product of odd primes, the problem with which we shall now be concerned, it will be shown below that indeed some general rules do apply. The one major difference with the prime moduli of the previous part, as shown below, is that in this case there are not primitive roots, and therefore that the principal group cannot encompass all the permitted integers. There is therefore constantly at least one secondary group. However, there are also striking analogies. For example, in mod. $p$  algebra,  $p$  prime, one cannot chose as modulus the base  $p$ . Analogously, if the modulus is composite,  $C = p_1 p_2 \dots p_n$ , the  $p_i$  being odd primes, the base  $\alpha$  cannot be one of the primes  $p_i$ , and all the multiples of the  $p_i$  are excluded from the period. (See however Part D.) As a consequence, a parameter of major interest is the quantity  $F = \prod (p_i - 1)$ , which describes the number of permitted residues. In the simplest case of  $C = p_1 p_2$ , the number of permitted residues is  $C - (C/p_1 + C/p_2) + 1$ , the +1 coming from the fact that the last term  $p_1 p_2$  is deleted twice. This is equal to  $(p_1 - 1)(p_2 - 1)$ , and the relationship generalizes to the product of any number of the first powers of odd primes. (See Appendix I).

As for prime moduli, there are here also open and closed groups.

A point of interest is the following : while the period for prime moduli can only be determined though trial and error, that of composite moduli may be predicted, at least in principle, from the periods of the component primes.

The analysis to follow is based on the two following number properties : a) Fermat's theorem ; b) The property that if  $\alpha^n$ ,  $n$  integer, equals  $+1 \pmod{p_1, \pmod{p_2, \dots \pmod{p_r}}$ , then

$$\alpha^n \equiv +1 \pmod{p_1 p_2 \dots p_r} \quad (6)$$

This congruence is a direct consequence of Eratosthenes' sieve, when applied to the powers of  $\alpha$ . If powers  $q_i$  of the  $p_i$  enter into the definition of  $C$ , the congruences  $\alpha^n \equiv +1 \pmod{p_i^{q_i}}$  should also be respected. The converse property is true : if  $\alpha^n \equiv +1 \pmod{p_1 p_2 \dots p_r}$ , then  $\alpha^n \equiv +1 \pmod{p_1, \pmod{p_2, \dots \pmod{p_r}}$ . The relationship is not true if  $\alpha^n$  is not  $\equiv +1$ , and then congruence to unity is not achieved for at least one of the primes entering in  $C$  at least one of the primes  $p_i$ .

The above will be detailed in what follows. For simplicity, it will often be assumed that  $C$  is the product of two or three odd primes. Generalization to  $r$  distinct primes is straightforward.

In what follows no distinction will be made between the residues and indices of the principal group and the “residues” of secondary groups. All indices and rests of division will be called indices and residues.

**B.1. The modulus  $C$  is the product of  $r$  odd primes  $p_i$ , and the base  $\alpha$  is a primitive root of all  $p_i$ .**

Let us begin with  $C$  being the product of two primes  $p_1$  and  $p_2$ . Here the number of permitted residues is  $F=(p_1-1)(p_2-1)$ . Since  $\alpha$  is by hypothesis a primitive of both  $p_1$  and  $p_2$  one has by Fermat's theorem :

$$\alpha^{p_1-1} \equiv 1 \pmod{p_1} \quad (7a)$$

$$\alpha^{p_2-1} \equiv 1 \pmod{p_2} \quad (7b)$$

and

$$\alpha^{(p_1-1)/2} \equiv -1 \pmod{p_1} \quad (8a)$$

$$\alpha^{(p_2-1)/2} \equiv -1 \pmod{p_2} \quad (8b)$$

One can elevate (8a) to the power  $(p_2-1)$  and (8b) to the power  $(p_1-1)$  to obtain

$$\alpha^{(p_1-1)(p_2-1)/2} \equiv +1. \quad (9)$$

since both  $p_1-1$  and  $p_2-1$  are even. Because of (7a), (7b), the relationship holds mod  $p_1$  as well as mod  $p_2$ , and considering (6) above, the relationship holds also mod  $p_1p_2$ .

As a result, for  $C=p_1p_2$ , *the period stops and recurrence occurs at most for the index  $(p_1-1)(p_2-1)/2$ , never for the index  $(p_1-1)(p_2-1)$* . Thus the modulus  $C$ , the product of two odd primes, *has no primitive roots*. The principal group Gr1 contains  $(p_1-1)(p_2-1)/2$  elements among the permitted integers and there is therefore a secondary group Gr2, which will contain those integers which are not residues of Gr1. To find the residues in Gr2, one should proceed as in the case of prime moduli. Here again one will find *open* and *closed* groups.

Examples: take  $C = 3 \times 5 = 15$ , base 2. One has, in enum.1, the period [2,4,8,1] forming the principal group Gr1, and the period [7,14,13,11] forming the secondary group Gr2. Multiples of 3 and 5 cannot of course be residues. The groups are *open*. In open groups the residue  $C-1 = p_1p_2-1$  never pertains to the principal group, since this is the characteristic property of *closed* groups. It is to be found into the secondary group, at some undetermined index. Take now  $C = 3 \times 11 = 33$ ; the principal group Gr1 in base 2 is [2,4,8,16,32,31,29,25,17,1] and the secondary group Gr2 is [5,10,20,7,14,28,23,13,26,19]. The groups are *closed*, and the residue  $p_1p_2-1$  pertains to the index  $(p_1-1)(p_2-1)/4 = 5$  of Gr1.

Though the period cannot exceed  $F/2$ , it can be shorter. Example : take  $C=5 \times 37=185$ ,  $F=144$ ,  $F/2=72$ ,  $F/4=36$ . Writing down the period base 2, one finds that it has 36 terms, that is it is equal to  $F/4$ .

The above can be generalized for  $C=p_1p_2 \dots p_r$ . For example, if one takes  $r=3$ , then  $F=(p_1-1)(p_2-1)(p_3-1)$  and a reasoning analogous to the one above will permit to find that

$\alpha^{(p_1-1)(p_2-1)(p_3-1)/4} \equiv +1$ , so that the period of a triple product cannot exceed  $F/4$ , though it can be shorter. Thus, here there are at least four groups, and generally, if  $C=p_1p_2 \dots p_n$ , the maximum period is at most equal to  $\prod(p_n-1)/2^{n-1}=F/2^{n-1}$ .

Whether the period is exactly  $F/2^{n-1}$  or less, is a problem examined below. As an example, the period base 2 of  $C=3 \times 5 \times 11=165$ , with  $F=80$ , is of  $F/4=20$  terms, while that of  $C=3 \times 5 \times 13=195$ , with  $F=96$  terms, is of  $F/8=12$  terms. The reason for this difference is that  $C=195$  divides  $2^{F/8}-1$ , while  $C=165$  divides  $2^{F/4}-1$ , but not  $2^{F/8}-1$ . (See the next section and Appendix II.)

As was the case for prime moduli, it does not seem that there exists any rule, that will permit to predict whether the groups of a composite modulus  $C$  are *open* or *closed*. One has to write down the periods to check. However, as above and at least for small values of  $C$ , it seems that the groups are more often open than closed. The above results can be summarized in the following theorem:

**THEOREM:** *When besides the principal there are also secondary groups, as is necessarily the case for composite moduli, these groups are either all closed or all open.*

## **B.2. The modulus $C$ is the product of $r$ odd primes at the first power, and the base $\alpha$ is a primitive root of none of the component primes.**

Taking again the simplest case of  $C=p_1p_2$ , one can write

$$\alpha^{(p_1-1)/D_1} \equiv 1 \pmod{p_1} \quad (10a)$$

$$\alpha^{(p_2-1)/D_2} \equiv 1 \pmod{p_2} \quad (10b)$$

As in the previous case we can rise (10a) to the power  $(p_2-1)/D_2$  and (10b) to the power  $(p_1-1)/D_1$  to obtain  $\alpha^{(p_1-1)(p_2-1)/D_1D_2} \equiv 1 \pmod{p_1}$  and  $\pmod{p_2}$ . Referring again to relationship (6), one also has  $\alpha^{(p_1-1)(p_2-1)/D_1D_2} \equiv 1 \pmod{p_1p_2}$ . The period is  $P=(p_1-1)(p_2-1)/D_1D_2$  and it is the longest possible, at least when  $D_1$  and  $D_2$  are relatively prime. For, in the latter case, if we divide  $P$  by say  $q$  an integer greater than 1, this  $q$  should either divide  $(p_1-1)/D_1$  or  $(p_2-1)/D_2$ , which is impossible, if one wants relationships (10) to be preserved. The same considerations can be extended to  $C=p_1p_2 \dots p_r$ , with  $r$  integer larger than 2.

Examples. Take  $C=7 \times 17=119$ , base 2. Then  $F=6 \times 16=96$ ,  $F/2=48$ ,  $F/4=24$ . Here  $D_1=D_2=2$ , so that the period should equal (at most)  $96/4=24$ . One indeed finds for the principal group the period  $[2,4,8,16,32,64,9,18,36,72,25,50,100,81,43,86,53,106,93,67,15,30,60,1]$ . If instead one considers  $C=7 \times 43=301$ , then  $F=252$  with  $D_1=2$  and  $D_2=3$ , so that the periods could have *a priori*  $F/6=42$  terms. One checks that this is indeed the case.

But consider now  $C=2047=23 \times 89=2^{11}-1$ , with  $F=22 \times 88=1936$ . Since it is found that  $D_1=2$  and  $D_2=8$ , normally the period should have been of  $F/16=121$  terms. But it is evident that the period is in fact of only 11 terms, since  $2^{11}=2048$ . One observes however that  $22=11 \times 2$  and  $88=11 \times 8$ , so that  $D_1D_2$  can be multiplied by 11, leading to a period of 11 terms and 121 associated groups.

## **The unmistakable procedure to predict the length of the period for a given composite modulus.**

As shown above, some ambiguities may exist for the prediction of the length of the period for a given composite modulus  $C$  and an arbitrary base  $\alpha$ , if the standard methods indicated above are used. Therefore it is convenient to dispose of a method devoid of any ambiguities and valid for all situations, whether all the components of  $C$  admit  $\alpha$  as a PR, none of them, or part of them.

In this respect use shall be made of relationship (6). Let  $C = p_1 p_2 \dots p_r$ , and  $F = (p_1 - 1)(p_2 - 1) \dots (p_r - 1)$ . Compute all the divisors  $\delta_i$  of  $F$  such that  $F/2^{r-1} > F/\delta_i > C$ . Then compute the integers  $A$  corresponding to the powers  $F/\delta_i$  and find, in for example a descending way for the  $\delta_i$ 's, the least divisor  $\delta(\min)$  for which the integer  $A = \alpha^{F/\delta(\min)}$  is congruent to unity mod. $C$ . According to (6), as the integer  $A$  has a residue  $+1$  mod. $(C = p_1 p_2 \dots p_r)$  it will also have a residue  $+1$  for all the  $p_i$ 's. The period shall stop and recurrence shall be initiated at the index  $F/\delta(\min)$ . Notice that here one may have  $p_i = p_{i+1} = p_{i+2} \dots$  to take account of the powers involved in the primes defining  $C$ .

The method, certainly safe and of general validity, presents however an inescapable drawback: as soon as  $F$  is large enough (and especially when the base  $\alpha$  is not chosen among the first few integers), the exponentiation may lead to numbers so large that they will elude the reach of even the most powerful computers in use. Therefore, in addition to this unmistakable and of general validity method, one may have, as a first approach, to examine the alternative method based on the value of  $F/D$ , as this does not imply integers larger than  $F$ .

**B.3. The modulus  $C$  is the product of two or more odd primes  $p_i$  at the first power, and the base  $\alpha$  is not a primitive root for some of the components of  $C$ .**

This case is slightly more intricate than the two previous cases, and this is why it has been left last. As usual, we begin with the simplest case of  $C$  being the product of two odd primes, the base  $\alpha$  not being a PR for one of these primes. Quite generally, one has

$$\alpha^{p_1-1} \equiv 1 \pmod{p_1} \quad (11a)$$

$$\alpha^{(p_2-1)/D} \equiv 1 \pmod{p_2} \quad (11b)$$

where  $D$  is an integer such that  $(p_2-1)/D$  is the period mod. $p_2$ . As previously, one can raise (11a) to the power  $(p_2-1)/D$ , and (11b) to the power  $p_1-1$  to obtain

$$\alpha^{(p_1-1)(p_2-1)/D} \equiv 1 \pmod{p_1}$$

$$\alpha^{(p_1-1)(p_2-1)/D} \equiv 1 \pmod{p_2}$$

and consequently, because of (6), also

$$\alpha^{(p_1-1)(p_2-1)/D} \equiv 1 \pmod{p_1 p_2} \quad (12)$$

$(p_1-1)(p_2-1)/D$  gives the “standard” period when  $\alpha$  is not a PR of one of the primes. But the period may be shorter (see below).

Examples. Take  $C = 5 \times 7 = 35$ ,  $F = 24$ ,  $\alpha = 2$ ,  $\alpha$  is not PR of 7. One finds in enum.1 two groups of twelve residues each :

and

$$[2,4,8,16,32,29,23,11,22,9,18,1] \quad \text{mod.35} \quad (\text{principal group})$$

$$[6,12,24,13,26,17,\underline{34},33,31,27,19,3] \quad \text{mod.35} \quad (\text{secondary group})$$

The groups are *open*, since the residue  $C-1 = 34$  is found in the secondary group at the index seven. The sum of each element of index  $v$  of Gr1 when added to the element of the same index of Gr2 sums up to 35, if the index 7 of 34 in Gr2 is circularly pushed as to occupy the position of the index 12, previously being the index of 3.

Take now  $C = 3 \times 43 = 129$ ,  $F = 84$ ,  $\alpha = 2$ ,  $\alpha$  not PR of 43. Here one finds in enum.1 that there are in all six groups of fourteen elements each, while relationship (12) states that there should be three groups of 28 elements each, but as emphasized this is only the largest possible period. Therefore, whenever possible, one should predict the actual length of the period using the *unmistakable method* indicated in B.2. The principal group Gr1 has the following residues :

$$[2,4,8,16,32,64,\underline{128},127,125,121,113,97,65,1] \quad \text{mod.129} \quad (\text{principal group})$$

Obviously here the groups are *closed* and the element  $C-1$  of index 7 of the principal group when added to unity of index 14 yields  $C = 129$ . More specifically, the sum of two residues of respective indices  $v$  and  $v+7$  yields  $C = 129$ . The detailed residues of all six groups will not be given here, since this is not of great interest.

For  $C$  being the product of  $r$  primes  $p$  which admit the base  $\alpha$  as a PR, and  $s$  primes  $q$  which do not, one can tentatively write as follows for the maximum possible period:

$$P_{\max} = 1/2 \prod_{i,j} [(p_i-1)/2^{r_i-1}] (q_j-1)/D_j] \quad (13)$$

#### B.4. The modulus $C$ is the product of the powers of two or more odd primes $p_i$ .

What now happens when  $C$  is of the form  $C=(p_1)^u.(p_2)^v$  with at least one of the integers  $u,v$ , being larger than one? Let us assume first that the base  $\alpha$  is a PR of both the components of  $C$ . Obviously, one should compare this case with the simpler case where one has as above  $C = p_1.p_2$ . In the latter case as already shown the number of permissible residues is  $(p_1-1)(p_2-1)$ . Taking the simplest case of only two groups, there will be  $(p_1-1)(p_2-1)/2$  terms in each group. Assume now that  $C$  is of the form  $C=(p_1)^2.p_2$ . The number of permitted residues is clearly  $(p_1)^2.p_2 - [(p_1)^2.p_2]/p_1 - [(p_1)^2.p_2]/p_2 + 1$ , the  $+1$  coming from the fact that the term  $(p_1)^2.p_2$  has been deleted twice. This is  $p_1(p_1-1)(p_2-1)$  and thus, each period will have at most  $p_1(p_1-1)(p_2-1)/2$  terms.

Example. Let  $C=3^2 \times 5=45$ , base 2. The permitted residues are  $F = 3 \times 2 \times 4=24$ , so that the period of each group should be 12. Writing down the periods one finds indeed :

$$[2,4,8,16,32,19,38,31,17,34,23,1] \quad (\text{principal group})$$

$$[7,14,28,11,22,\underline{44},43,41,37,29,13,26] \quad (\text{secondary group})$$

The groups are *open*, and the residue 44, is found at the index six of the secondary group.

The above are of general validity: if  $C=(p_1)^u.(p_2)^v$ , then the number of permitted residues is  $(p_1)^{u-1}.(p_2)^{v-1}(p_1-1)(p_2-1)$ . If there are  $N$  distinct groups, the period of each, and especially the principal, will have  $(p_1)^{u-1}.(p_2)^{v-1}(p_1-1)(p_2-1)/N$  terms.

The above generalize to  $C$  being the product of any number of odd primes for which the base is a PR. If only first powers are involved, by a reasoning analogous to that given in **B.**, the maximum period  $P$  of the product of  $r$  primes will be

$$P_{\max} = \prod_{i=1}^r (p_i - 1) / 2^{n-1} \quad (14a)$$

If powers  $u_i$  of the  $p_i$  are also involved, the maximum period will be

$$P_{\max} = \prod_{i=1}^r p_i^{(u_i-1)} (p_i - 1) / 2^{n-1} \quad (14b)$$

In the latter case of a maximum period the total number of groups will be  $2^{n-1}$ , that is as usual the principal and  $2^{n-1}-1$  secondary groups.

Examples: Take  $C=3 \times 5 \times 7=105$ . Then the maximum period will be  $F/4=2 \times 4 \times 6/4=12$  terms. One indeed finds for the principal group base 2 the period:

$$[2, 4, 8, 16, 32, 64, 23, 46, 92, 79, 53, 1], \text{ mod. } 105.$$

The maximum period is here achieved. The groups are *open*, and the maximum residue 104 is to be found in the secondary group Gr13 at the index 4.

Take now  $C=3^2 \times 5 \times 7=315$ . The maximum possible period will here be  $F=3(3-1)(5-1)(7-1)/4=144/4=36$ , while the least possible will display at least  $x$  integers, where  $2^x > 315$ . However, it is found that the period is in fact again twelve, that is  $144/12$ , so that the maximum period is not here achieved. The principal group is:

$$[2, 4, 8, 16, 32, 64, 128, 256, 197, 79, 156, 1], \text{ mod. } 315$$

The groups are open, and the residue 314 is found in the secondary group Gr59 at the index 5.

The rationality of these distinct behaviours between  $C=3 \times 5 \times 7$  and  $C=3^2 \times 5 \times 7$  is explained in section C.2 and Appendix II.

In the general case where  $C=p_1 p_2 \dots p_n q_1 q_2 \dots q_s$ , the  $p_i$  admitting the base  $\alpha$  as PR and the  $q_j$  not, one can tentatively write for the maximum possible period :

$$P_{\max} = (1/2) \prod_{i,j} [p_i^{(u_i-1)} (p_i - 1) / (2^{r-1})] [q_j^{(u_j-1)} (q_j - 1) / D_j] \quad (15)$$

Because of the ambiguity of the above relationship, one should apply whenever possible the *unmistakable method* of B.2. If the numbers involved are too large to be managed by your computer, well, ... then write a computer software ordering the computer to write down the period, whatever its length !

## Part C. Even composite moduli.

One in two integers is an even number. Therefore it is necessary to examine also the case of even composite moduli, so the more that a physicist interested in deterministic mapping, as suggested in the introduction, is free to choose in his mod.1 congruences as denominators even integers, for example  $D=2^3 \times 5=40$  or  $D=2 \times 7^2 \times 13=1274$ . In what follows we first examine the case of even composite moduli of the form  $2^N$ , and subsequently the general case of composite moduli of the form  $2^N \beta^\xi \gamma^\psi \dots$ , where  $\beta, \gamma$  etc. are odd primes. The

definition of the parameter F of the previous case remains valid. For example, for  $C=2^4 \times 11=176$  will correspond the  $F = 2^3 \times 10=80$ .

### C.1 The modulus C is of the form $2^N$ .

One should distinguish two cases : two is PR of the base  $\alpha$ , and two is not a PR of the base  $\alpha$ .

#### C.1.1 Two is a primitive root of the base $\alpha$ .

Here the base  $\alpha$  is necessarily an odd prime. Trials taking small values of  $\alpha$ ,  $\alpha=3, 5, 11, \dots$  and varying the exponent N, suggested the following relationship, when 2 is a PR of the base  $\alpha$  :

$$\alpha \exp 2^{N-2} \equiv 1 \pmod{2^N} \quad (16)$$

A proof of this relationship can be given by induction : first it is known<sup>24</sup> that for any odd integer  $\beta$ , one has  $\beta^2 \equiv 1 \pmod{2^3=8}$ . Thus, (16) is true for  $N=3$ , whatever the value of  $\alpha$ , with 2 being a PR of  $\alpha$ . To prove (16), it suffices to establish that if (16) is true for N, it is also true for  $N+1$ . For this, let us square both sides of this relationship :

$$[\alpha \exp 2^{N-2}]^2 = \alpha \exp 2^{N-1} = 1$$

However, when squaring the exponent in  $\alpha$ , one also automatically doubles the period. Therefore, the above relationship is true  $\text{mod. } 2 \times 2^N = \text{mod. } 2^{N+1}$  and thus one can write

$$\alpha \exp 2^{N-1} \equiv 1 \pmod{2^{N+1}} \quad (16')$$

which provides the proof of (16) for any N, by substitution of  $N'=N+1$  in (16'). This result was known to Gauss who first proved it.<sup>25</sup>

Relationship (16) implies that the period is  $2^N/4 \pmod{2^N}$ ; since even integers are automatically excluded from the period, the period could have been equal to  $N/2$ . However, it was not *a priori* evident that the period is only  $N/4$ . Since however the correct period  $\text{mod. } 2^N$  is  $2^N/4$ , it follows that there are always two groups, the principal and the secondary group.

Example. Let the modulus be  $2^5=32$  and the base  $\alpha=5$ . 2 being a PR of 5, there should be two groups of eight terms each. One finds indeed,

$$[5, 25, 29, 17, 21, 9, 13, 1] \quad \text{principal group}$$

and

$$[3, 15, 11, 23, 19, 31, 27, 7] \quad \text{secondary group}$$

One immediately checks that all odd integers from 1 to 31 are covered. One also remarks, in this particular case (an unexpected result), that the succession of these ordered odd integers is to be found alternatively in the principal and the secondary groups: one finds



in the principal group the odd integers 1,5, 9, 13, 17, 21, 25, 29, and in the secondary 3, 7, 11, 15, 19, 23, 27, 31. In other cases, the alternation goes two by two, as for the base 3, mod.2<sup>5</sup> :

[3, 9, 27, 17, 19, 25, 11, 1]      principal group

and

[5, 15, 13,7, 21, 31, 29, 23]      secondary group

Other bases alternating one by one are  $\alpha=13, 29, 37 \dots$ , while others alternating two by two are  $\alpha=3, 11, 19, \dots$  Base 31 is special, because  $31=2^5-1$ . Since unity is always to be found in the principal group, and the last residue  $2^N-1$  in the secondary, it follows that the groups are necessarily *open*. That  $2^N-1$  appears in the secondary group is almost evident ; for, mod.(2<sup>3</sup>=8), whether the alternation is one by one or two by two,  $7=8-1$  appears manifestly in the secondary group. If now the modulus is  $2^4=16$ , one has to put side by side in succession two quartets of odd integers. And so on. We don't know if there are alternations of more than two successive odd residues. If there are, they should be of the form  $2^{N-c}$ , c integer < N, in order that they divide  $2^N$ . Whatever the case, one can safely enunciate : *if two is a PR of the base  $\alpha$ , the two groups mod.2<sup>N</sup> are never closed.*

One final remark is as follows : if one adds index  $v_1$  of the principal group and  $v_2$  of the secondary group, one obtains the residue  $v_1+v_2$  of the secondary group. (If  $v_1+v_2$  exceeds the number of terms  $v$  in the period, one has to subtract  $v$  from the sum.) This also is evident, since equal values of  $v_1+v_2$  correspond to the same integer in non modular algebra, and therefore to the same residue in mod.2<sup>N</sup> algebra.

This curious phenomenon of alternation is challenging, however, not being an essential feature of recurrences, it shall not be any further examined in this paper.

### C.1.2 Two is not a primitive root of the base $\alpha$ .

If now one looks for the principal period of base 7 mod.2<sup>3</sup>, which corresponds to  $7\exp 2^{N-2} = 7^2$  one finds the expected period of two terms [7,1] which as above equals  $2^{N-2}$  ; now, mod.(2<sup>5</sup>=32), one has  $7\exp 2^{5-2} = 7^8 = 5\,764\,801 \equiv 1 \pmod{32}$  as expected, and the period should have been of eight terms; however, one finds that  $7^4=2401 \equiv 1 \pmod{32}$  also, and that the period displays only the four terms [7,17,23,1], being now equal to  $2^{N-3}$ . The same phenomenon occurs for  $\alpha = 23$ , and 41, but for  $\alpha=17, 47$  and 71, the period is  $2^{N-4}$ .

One should perhaps be able to demonstrate that from the couple of congruences

$$2^{(\alpha-1)/q} \equiv 1 \pmod{\alpha} \quad (17a)$$

$$\alpha^{N-c} \equiv 1 \pmod{2^N}, \quad (17b)$$

$q$  and  $c$  integers,  $q$  dividing  $\alpha-1$ , one can establish a relationship between  $q$  and  $c$ , so that for  $q=1$ ,  $c=2$ , and for  $q>1$ ,  $c>2$ .

CONJECTURE: In the absence however of the proof for such a correlation, the conjecture is here made that *the length of the period is  $2^{N-2}$  when 2 is a PR of  $\alpha$  and  $2^{N-3}$  or less when 2 is not.*<sup>26</sup>

As previously in C.1.1., the ordered odd integers are distributed in regular patterns among the groups, but now these patterns are more complicated.

## C.2 The modulus is of the form $2^N \beta^\xi \gamma^\psi \dots$ , or $\beta^\xi \gamma^\psi \dots$ , where $\beta, \gamma, \dots$ are odd primes.

Let us first come back to the examples of section B.2, where it was found that for  $C=(3 \times 5 \times 7)=105$  and  $C=(3^2 \times 5 \times 7)=315$ , the periods base 2 had the equal length of twelve terms, corresponding respectively to  $F/4$  and  $F/12$ . However, when one tries  $C=(3^3 \times 5 \times 7)=945$ , the period is three times as large, i.e. of thirty six terms, corresponding again to  $F/12$ . For  $C=2^N \times 17$  on the other hand, the period base 7 remains equal to sixteen terms, from  $N=0$  (that is mod.17) up to  $N=6$ , that is mod.1088. Then the period doubles to thirty two terms for  $N=7$ , i.e. mod.2176. As a final example, for  $C=2^N \times 11$ , base 3, for  $N=1$  the period is of five terms, then for  $N=2,3$  and 4 of ten terms, increases to forty terms for  $N=5$  and goes to eighty terms for  $N=6$ .

At first sight there is no rationality in these examples. One can make first the following observation : Let the penultimate term of index  $v$  of the period base  $\alpha$  and mod. $m^k C$  be  $T$ , where  $m=2, \beta, \gamma, \dots$  is the integer defining the increase of the modulus when going from one modulus to the next one. Suppose that  $m^k C$  is such that  $\alpha T \text{ mod. } m^k C \equiv 1$ . For this modulus the period will stop and the index  $v+1$  and recurrence will be initiated at the index  $v+2$ . Let now the next modulus be  $m^{k+1} C$ , and the term of index  $v$  again be  $T$ . If now the term  $\alpha T$  of index  $v+1$  is less than the modulus  $m^{k+1} C$ , the period will not stop at the index  $v+1$  but will continue : the period is increased.

The above observation is however of very limited predictive power. The rationality of the question is again found using the *unmistakable procedure* of section B.2. Let for the modulus  $m^k C$  the period be determined by the associated parameter  $F_k/\delta_k(\min)$ , which involves that  $\alpha^{F_k/\delta_k(\min)} \equiv 1 \text{ mod. } m^k C$ . As long as  $\alpha^{F_k/\delta_k(\min)} \equiv 1$  with respect to the moduli that follow, that is mod. $m^{k+1} C$ , ... mod. $m^{k+p-1} C$ , the period length remains unchanged. If however at mod. $m^{k+p} C$  the congruence to unity is no longer achieved, then the period is increased by a factor of  $m$ . For, in this case, congruence to unity is achieved at the new index  $F_{k+p}/\delta_{k+p}(\min)$ . The examples given above are being detailed in this respect in Appendix II.

## PART D. The modulus is composite, and the base is one of the factors of the modulus.

Prime moduli cannot of course display periods and recurrences, when the base is the same as the modulus. If for instance one takes as base 5 and modulus  $5^4 = 625$ , the series will stop after four terms, the last one being  $\equiv 0 \text{ mod. } 5^4$ . However, when  $C$  is the product of two primes at their first power,  $C=p_1 p_2$ , and only in this case, a normal process of recurrence occurs, when one of the primes is taken as basis. To show this, let  $p_2 > p_1$ , and let  $x = p_2 - p_1 > 0$ . If one chooses  $p_1 = \alpha$  as the base (to maintain the Greek symbol used throughout for the base), and if further it is first assumed that  $\alpha$  is a PR of  $p_2$ , one can write as follows the residues:

$$\alpha, \alpha(p_2 - x), \alpha(p_2 - x)^2, \dots, \alpha(p_2 - x)^{p_2-1} \quad \text{mod. } \alpha p_2$$

Developing the powers of  $(p_2 - x)^t$ ,  $1 \leq t \leq p_2 - 1$ , it is easily seen that all the terms are  $\equiv 0 \text{ mod. } \alpha p_2$ , except for the last one, which is  $\alpha x^t$ , and that all these residues are different. Now, since  $\alpha$  is assumed a PR of  $p_2$ , and  $p_2$  does not divide  $x$ , one has from Fermat's theorem

$$x^{p_2-1} \equiv 1 \text{ mod. } p_2 \quad (18)$$

It is now permitted to multiply both terms of the congruence (18) by  $\alpha$  and at the same time also the modulus by  $\alpha$ , (because if two integers differ by one, their product by  $\alpha$  will differ also by  $\alpha$ ) to obtain :

$$\alpha x^{p_2-1} \equiv \alpha \pmod{\alpha p_2} \quad (19)$$

Consequently, at the index  $p_2$  the residue  $\alpha$  is recovered, and there is a period of  $p_2-1$  terms.

Example :  $\text{mod.}(51=3 \times 17)$ , base 3, 3 being a PR of 17, one finds the period of sixteen terms [3,9,27,30,39,15,45,33,48,42,24,21,12,36,6,18] and then back to 3. Conversely, one may choose as the base 17, leading to the period of two terms [17,34]. Of course, never unity appears as a residue, as both  $\alpha=p_1$  and  $p_2 > 1$ . This is the essential difference with the normal procedure where the base is not part of the primes entering C, and where there is always a principal group containing unity, absent from the secondary groups.

If now  $\alpha$  is not a PR of  $p_2$ , the period length will be  $(p_2-1)/D$ , D dividing  $p_2-1$ , with a total of D groups, the principal and D-1 secondary, the principal group being here defined as the one containing the base  $\alpha$ .

Example :  $\text{mod.}(34=2 \times 17)$ , base 2, 2 not being a PR of 17. Here there are two groups base 2 :

[2,4,8,16,32,30,26,18] principal    [6,12,24,14,28,22,10,20] secondary

In base 17 there is a period constituted by the single term 17. One checks that these residues cover exactly those integers which are forbidden residues in normal periods, i.e. those constructed taking as base a prime not entering the definition of the composite modulus C. (Except for the residue C, since this would put an end to the recurrence.)

The situation is more complex when more than two primes enter into the definition of C, or when C is the product of powers of primes. Analysis of such cases, though not devoid of interest, lies outside the scope of the present work.

## CONCLUSION.

As emphasized in the introduction, the incentive for writing this article originates in the recurrence properties of deterministic mapping, a subject lying in the frontier between physics and modern mathematics. The emphasis was put on periods and groups, rather than on congruences.

The above analysis is especially oriented towards the Bernoulli mapping. This mapping has been generalized to any basis and any modulus, instead of been restricted to basis two and modulus one. Put another way, any rational point inside the segment [0,1] may be taken as a starting point using any integer modulus. Such a detailed analysis is not known to the author to have been made elsewhere. Emphasis was put on secondary groups, as defined in the text. Such secondary groups are always present in composite moduli while in prime moduli they may or may not be present, depending on whether the base is a primitive root of the modulus or not. The secondary groups have been classified as *open* or *closed* secondary groups. Also, special attention has been given to composite moduli, including even composite moduli.

All propositions have been followed by numerical examples, so that the reader, whatever his mathematical status, may acquire a good and easy knowledge of the topic.

In retrospect, it appears that the only difference between the principal and the secondary groups lies in the fact that unity appears only in the principal group. (Except in the case where the base is one of the components of a composite modulus which is the product of

two primes, in which case unity is never present, neither in the principal nor in the secondary groups). If one is not interested in this difference, there is a perfect symmetry in the properties of the groups. The periods are the same, the groups are all *open* or all *closed* at the same time. In even moduli, the scheme representing the distribution of the permitted odd integers within the groups respects perfectly regular patterns. Though for the mathematician the presence of unity may be important, for the physicist interested in deterministic mapping, the presence or absence of unity in the period is not necessarily of outstanding interest. He can as well use, with the same success as far as recurrence is concerned, the principal or a secondary group. He can also use the recurrence of a composite modulus being the product of two primes, and take as the basis one of the primes.

In Annexes I and II below some points in the main text are further analysed. In Annexe III, an interesting recurrent mapping, Arnold's cat map, is briefly examined.

## APPENDIX I: The F parameter.

It was stated in **Part B**, that for composite moduli  $C = \prod p_i$ , where the  $p_i$  are the primes entering into the definition of  $C$ , (first assumed to be at the first power), the number of permitted residues was given by the parameter  $F = \prod (p_i - 1)$ . A proof of this in the general case may be given by induction.

Let us begin, as in Part B, with the simplest case where  $C$  is the product of two distinct primes,  $C_2 = p_1 p_2$ . To  $C_2$  should be subtracted as non possible residues the multiples of  $p_1$  and  $p_2$ , which are  $C_2/p_1 + C_2/p_2 = p_1 + p_2$ . To this however should be added unity, for the term  $p_1 p_2$  has been subtracted twice instead of once, so the number of permitted residues is  $p_1 p_2 - (p_1 + p_2) + 1$ . This is clearly equal to  $F_2 = (p_1 - 1)(p_2 - 1)$ .

Let us now multiply  $C_2$  by a third distinct prime  $p_3$ , so that the new modulus is  $p_3 C_2 = C_3 = p_1 p_2 p_3$ . Again we have to exclude as non possible residues all the multiples of  $p_1$ ,  $p_2$ ,  $p_3$ , which number  $C_3/p_1 + C_3/p_2 + C_3/p_3 = p_1 p_2 + p_1 p_3 + p_2 p_3$ . However, a number of these excluded residues has been counted twice. These are the multiples of the  $p_{ij}$ , whose number is  $C_3/p_1 p_2 + C_3/p_1 p_3 + C_3/p_2 p_3 = p_1 + p_2 + p_3$ , and this quantity should be added to the previous one. Finally, one should remark that the term  $p_1 p_2 p_3$  was counted three times instead of two as being a twice slashed non residue, so that  $-1$  should be added to the final result, which is therefore  $p_1 p_2 p_3 - (p_1 p_2 + p_1 p_3 + p_2 p_3) + (p_1 + p_2 + p_3) - 1$ . This is clearly equal to  $F_3 = (p_1 - 1)(p_2 - 1)(p_3 - 1)$ .

To continue, let us multiply  $C_3$  by a fourth distinct prime  $p_4$ , so that  $C_4 = p_4 C_3 = p_1 p_2 p_3 p_4$ . Following the same procedure as above, one must search for the non residues being the multiples of  $p_1$ ,  $p_2$ ,  $p_3$ ,  $p_4$ . These are  $C_4/p_1 + C_4/p_2 + C_4/p_3 + C_4/p_4 = p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4$ . However, among these terms, a number have been counted twice. These are all the terms  $C_4/p_{ij}$  ( $i \neq j$ ), which here are six, respectively  $p_{12}$ ,  $p_{13}$ ,  $p_{14}$ ,  $p_{23}$ ,  $p_{24}$ , and  $p_{34}$ , yielding the term  $-(p_{12} + p_{13} + p_{14} + p_{23} + p_{24} + p_{34})$ ; when so doing, however, some terms have been counted thrice instead of twice, and should therefore be subtracted to the partial result. These are all the terms of the form  $C_4/p_{ijk}$ , ( $i \neq j \neq k$ ), yielding the term  $-(p_1 + p_2 + p_3 + p_4)$ . Finally, the last permitted residue, i.e.  $p_1 p_2 p_3 p_4$ , has been deleted four times instead of three, so that unity should be added, and the final result is  $F_4 = p_1 p_2 p_3 p_4 - (p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4) - (p_{12} + p_{13} + p_{14} + p_{23} + p_{24} + p_{34}) - (p_1 + p_2 + p_3 + p_4) + 1$ . This is equal to  $F_4 = (p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)$ .

And so on, having constantly terms with alternating  $+$  and  $-$  signs. The last term is  $(-1)^n$ , so that when  $n$  is even unity should be added, and when  $n$  is odd, unity should be subtracted.

If now to  $C_n$ , ( $n=2, 3 \dots$ ), one or more component primes are in present in some power

$u, v, \dots, F_n$  is simply multiplied by the corresponding primes at the powers  $u-1, v-1, \dots$ . For, it is readily seen that in such cases all the terms in the above sums are also multiplied by  $u-1, v-1, \dots$ .

## APPENDIX II: Moduli of the form $2^N \beta^\xi \gamma^\psi \dots$ ,

In this Appendix we shall work out in more detail the examples given in section C.2.

1) As a first exercise let us consider the second example of section C.2, the modulus  $2^N \times 17$ , base 3. Since 3 is a PR of 17, the period of  $3 \bmod 17$  ( $\Rightarrow N=0$ ) has sixteen terms. Of course the period cannot be shorter for powers of  $N \neq 0$ . Thus for  $N=1$ , one finds  $\bmod(2 \times 17=34)$  the following period of sixteen terms :

[3,9,27,13,5,15,11,33,31,25,7,21,29,19,23,1]

Now the question arises up to what modulus (=which value of  $N$ ) the period keeps the length of sixteen terms ? The  $F$  parameter is here equal to  $2^{N-1} \times 16$ . Here however we can dispense ourselves from actually determining the values of  $F/\delta(\min)$  for each value of  $N$ , corresponding to the least value  $F/\delta(\min)$  for which  $\alpha^{F/\delta(\min)}$  is congruent to unity. We can use the following shortcut : let us first compute  $A=3^{16} = 43\,046\,721$  and let us divide  $A-1$  by the successive moduli when increasing  $N$ , i.e. 34, 68, 136, 272, 544, 1088 and 2176. All these divisions up to 1088, corresponding to  $N=6$ , yield integer numbers, meaning that the period keeps the value of sixteen terms. Division however by 2176 yields 19785,5, a non integer, meaning that  $3^{16}$  is not congruent to unity,  $\bmod 2176$ . To achieve congruence, the period has to be doubled to thirty-two terms, and this is verified by actually writing down the period.

2) For a second exercise consider now the case of  $C=3^N \times 5 \times 7$ , base 2. This example will be worked out in detail.

First consider  $C=3 \times 5 \times 7=105$ , with  $F=2 \times 4 \times 6=48$ . The divisors  $\delta_i$  of  $F$  are, in decreasing values, i.e. increasing period lengths, 24, 16, 12, 8, 6, 4 and 2. The divisors 24, 16, 12, and 8 are at once excluded, since these would lead respectively, for the last residues, to  $2^2=4$ ,  $2^3=8$ ,  $2^4=16$  and  $2^6=64$  which are less than the modulus 105. Let us then try the divisor 6,  $F/6=8$ ,  $2^8=256$ .  $F/6-1$  should be, if correct,  $\equiv 0 \bmod 105$ , that is 105 should divide 255. This is not the case, so that a period of eight terms is too short. Let us then try  $F/4=12$ , corresponding to a period of 12 terms ; now  $2^{12}=4096$ ,  $F/4-1=4095$ , which is divisible by 105, yielding 39. Thus  $\delta(\min)=4$ , and the period should be of twelve terms. One indeed finds the period,

[2,4,8,16,32,64,23,46,92,79,53,1],  $\bmod 105$ .

Consider now the modulus  $C=3^2 \times 5 \times 7=315$ , leading to  $F=3 \times 48=144$ . Since by increasing the modulus the period cannot become shorter, let us first examine the case of a period of twelve terms as above, corresponding to  $F/12 : F/12-1=4095$  should be divisible by 315, and this is the case,  $4095/315=13$ . One finds again a period of twelve terms :

[2,4,8,16,32,64,128,256,197,79,158,1],  $\bmod 315$ .

Let now  $C=3^3 \times 5 \times 7=945$ ,  $F=432$ . One can again try a period of twelve terms, corresponding to  $F/36$ .  $F/36-1=4095$ , but this is not divisible by 945. So the period should be larger, necessarily a multiple by three of the previous one.  $F/12=36$ ,  $2^{36}=68\,719\,476\,736$ , and one checks that  $2^{36}-1$  is divisible by 945, yielding 72 719 023. The corresponding period is, [2,4,8,16,32,64,128,256,512,79,158,316,632,319,638,331,662,379,758,571,197,394,788,631,317,634,323,646,347,694,443,886,827,709,473,1],  $\bmod 945$ .

Finally, consider  $C=3^4 \times 5 \times 7=2835$ , with  $F=1296$ . One may try again the period of thirty-six terms, in case the period length had remained unchanged. However, 2835 does not divide  $68\,719\,476\,735$ , so that the period should be trice as large, that is of 108 terms, corresponding again to  $F/12$ . If now one computes  $2^{108}=A$ , and divides this by 2835, one finds the integer, 114 468 625 629 074 683 168 661 735 653. Therefore 2835 divides  $A$ , and the period should have 108 terms. If he so wishes, the reader can check this for himself.

Of course, as already pointed out, if the exponentiation of the base leads to intractable numbers, and if consideration of the divisors of F leaves some doubts, the only solution would be to write a software directly computing the periods. However, the *unmistakable procedure* has at least the merit to explain the apparently irregular increase of the period when the exponents in the primes defining C are increased.

### ANNEXE III : Arnold's cat map and Arnold numbers and series.<sup>12,27</sup>

This deterministic chaotic map was devised as an example of a recurrent mapping by the Russian mathematician Vladimir I. Arnold (1935-...) in his book with A. Avez. *Ergodic Problems in Classical Mechanics*.<sup>12</sup> (See Wikipedia at the link *Arnold's cat map*.) Here follow succinct indications on this mapping.

Restricting ourselves in what follows to integer values for the variables x and y, Arnold's cat map is defined as follows:

$$x_{k+1} = 2x_k + y_k \quad y_{k+1} = x_k + y_k \quad (1a), (1b)$$

or, equivalently, in matrix notation :

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} \rightarrow \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}$$

(In fact with respect to ref.12, x has been substituted for y and vice versa. This however has no effect on what follows.)

Arnold's matrix  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

has a determinant one and is therefore unitary. It is invertible, the inverse matrix being

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

The two eigenvalues of the Arnold's matrix are  $\lambda_1 = (3+\sqrt{5})/2$  and  $\lambda_2 = (3-\sqrt{5})/2$ , so that  $\lambda_1 > 1$  and  $0 < \lambda_2 < 1$ . To  $\lambda_1$  corresponds an eigenvector in an expanding direction and to  $\lambda_2$  a perpendicular eigenvector in a contracting direction.

As remarked by Dyson and Falk<sup>27</sup> there is a simple relationship between the matrix generating Arnold integers and that generating the well known Fibonacci integers:<sup>28</sup>

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

that is  $F^2 = A$ . Also, from (1a) and ((1b) one easily finds that

$$x_{k+1} = 3x_k - x_{k-1} \quad \text{and analogously} \quad y_{k+1} = 3y_k - y_{k-1} \quad (2a), (2b)$$

Here one of the variables has disappeared, but by compensation two of the previous  $x_{k+1}$  or  $y_{k+1}$  are now needed to define the next term. However,  $x_{k+1}$  still defines the corresponding  $y_{k+1}$ , through  $y_{k+1} = x_{k+1} - x_k$

These numbers we shall call here to forth Arnold numbers  $a_i$ , so that  $a_i \equiv x_i$  in eq. (1a), in honor of the above indicated contemporary mathematician, V. I. Arnold, and correspondingly the succession of Arnold numbers, Arnold series. The definition of these  $a_i$  numbers presents the analogy with the definition of the Fibonacci numbers  $f_i$ , in that each  $a_i$  as each  $f_i$  are defined through the two previous terms in the series,  $a_{i-1}$ ,  $a_{i-2}$  and  $f_{i-1}$ ,  $f_{i-2}$ .

In modular algebra it is clear that both the Arnold A and Fibonacci F matrices are periodic. For, in eqs (1a) and (1b) mod.M there are  $2M(M-1)/2 = M(M-1)$  couples of different integers  $x_j$ ,  $y_k$  to which should be added M-1 couples of equal integers  $x_j$ ,  $y_i$ , the identical couple 0, 0 being forbidden. The total sums up to  $M^2 - 1$  possible couples. It follows that at most after  $M^2 - 1$  steps some couple of integers which has already appeared should appear again, and so a new period initiated. If it is assumed that there are no closed loops after a number of initial steps which do not repeat themselves, the new period shall begin with the first two terms initiating the process. In fact, the periods are much shorter than  $M^2 - 1$ . In ref. 27 it has been shown that the period cannot exceed  $3M$ .

It may be demonstrated that if the Arnold matrix A has a periodicity P mod.M, then the periodicity of a couple of numbers  $x_k$ ,  $y_k$  initiating Arnold's mapping, or equivalently the periodicity of the unitary vector originating in  $x_k$ ,  $y_k$  on a bidimensional grid on which acts the matrix (1) is also P. If this were not so, the length of the trajectories on the grid would be dependent on  $x_k$ ,  $y_k$ , and an asymmetry introduced in the problem which would completely change the periodicity, and consequently the reappearance of the cat. Analogous reasoning holds for the periodicity of the Fibonacci series in modulo M algebra. However, for a same modulus M, the Arnold and Fibonacci periods are not the same.

The proper Fibonacci series begins with 1, 1. In the same manner one can define a proper Arnold series by putting in (1a) and (1b)  $x_1=1$  and  $y_1=1$ . However, one may also consider a generalized Arnold series, in analogy with the generalized Fibonacci series, where  $x_1$ , and  $y_1$  may be any integers. Considering such generalized Arnold series and taking into account eq. (2a), the following proposition is evident: *Any integer number may be an element of an Arnold series, and this in an infinity of ways.*

Example: Taking  $x_1=a_1=1$  and  $x_2=a_2=1$ , the twenty first Arnold numbers of the proper Arnold series are, according to relation (2a):

1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10 946, 28 657, 75 025, 196 418, 514 229, 1 346 269, 3 524 578, 9 227 465, 24 157 817.

The period mod.5 is: 1,1,2,0,3,4,4,3,0,2, that is  $P=2M$ ; it is easily checked that the period mod.7 has eight terms, that is  $P=M+1$ , and mod.11 five terms, that is  $(M-1)/2$ .

Using (2a) it is found that

$$a_k a_{k+2} - (a_{k+1})^2 = a_{k-1} a_{k+1} - a_k^2 = \text{constant} \quad (3)$$

whatever k. Therefore, from the above three first terms of the proper Arnold series one has

$$a_{k-1} \cdot a_{k+1} - a_k^2 = 1 \quad (\text{for the proper Arnold series}) \quad (4)$$

whatever k. Of course, if a different Arnold series is considered, (3) will take other values. For example, if the two first terms are  $a_1 = 3$ ,  $a_2 = 7$ , leading to the series 3, 7, 18, 47, 123, 322, 843..., one finds from the three first terms of the series that relationships (3) equal

now 5.

Notice that to obtain Arnold series differing from the proper one, one should choose initial integers which are different from two successive integers appearing in the proper series.

Using relationships (2) and (4), one also finds that

$$a_{k+1}a_{k-2} - a_k a_{k-1} = \text{constant} \quad (5)$$

whatever k, the constant being equal to 3 for the proper Arnold series.

Quite certainly other such relationships can be found, however we shall be content here with the two examples (3) and (5) given above.

As a final remark, notice that three successive Arnold numbers are mutually coprime, because the *sum or difference of two coprime integers never have common decomposition factors with these two coprimes*.<sup>29</sup> From this and eqs (2a), (2b) it follows that no Arnold number is divisible by three and from  $y_{k+1} = x_{k+1} - x_k$  it follows that the x and y series are always distinct.

## REFERENCES

- 1) The Belousov-Zhabotinsky reaction is a far from thermodynamic equilibrium oscillating chemical reaction, discovered during the nineteen fifties by Belousov and then rediscovered by Zhabotinsky; see for example G. Nicolis and I. Prigogine, *Exploring Complexity*, Freeman, New York 1989; see also ref. 5.
- 2) Phase space: If a physical system has X degrees of freedom (parameters defining it), the state of this physical system at time  $\tau$  is described in an X dimensional reference frame by a point in that frame, and the successive states by a trajectory in the frame. For example, the frictionless pendulum is defined by the angle  $\theta$  with the vertical at time  $\tau_0$  and the corresponding angular momentum by  $d\theta/d\tau$ . There are two degrees of freedom, the phase space is defined in a two dimensional reference frame, and the trajectory for small oscillations in Cartesian coordinates  $\theta, d\theta/d\tau$ , is an ellipse.
- 3) Henri Poincaré, *Sur le problème des trois corps et les équations de la dynamique*, Acta Mathematica, 13, 1-270 (1890). See any text on ergodic theory, for example ref. 12.
- 4) Herzprung-Russel diagram: see the articles for white dwarfs, neutron stars and black holes in Wikipedia.
- 5) Excellent as an introduction to chaos theory is the following book : Pierre Bergé, Yves Pomeau and Christian Vidal, *L'ordre dans le chaos*, Hermann, Paris 1988. (In French.)
- 6) The autocorrelation function  $C(\tau)$  is defined by

$$C(\tau) = (1/T) \int_0^T X(t)X(t+\tau)dt$$

$\lim_{T \rightarrow \infty}$

where  $X(t)$  is a function of time  $t$ ,  $T$  the limits of integration and  $\tau$  the advance time. For periodic systems  $C(\tau)$  is periodic, while for chaotic systems  $C(\tau)$  tends to zero with increasing  $\tau$ .

7) This reasoning may satisfy the mathematician. However, in physics, the reasoning does not apply, because of Heisenberg's indeterminacy principle which excludes the exact knowledge of both the position and the velocity of a gas molecule at a given time.

8) Edward Lorenz, J.Atmos.Sci., 20,130 (1963); see also ref. 10.



- 9) H.G. Schuster, *Deterministic Chaos*, VGH Publishers, Berlin 1989.
- 10) Ian Stewart, *Does God Play Dice ? The Mathematics of Chaos* p.141.
- 11) For the Bernoulli map, see for example D.J. Driebe, *Fully Chaotic Maps and Broken Time Symmetry*, Kluwer Academic Publishers, Dordrecht 1999.
- 12) V. I. Arnold; A. Avez, *Problèmes Ergodiques de la Mécanique Classique*. Paris: Gauthier-Villars, 1967.; English translation: V. I. Arnold; A. Avez, *Ergodic Problems in Classical Mechanics*. New York: Benjamin, 1968.
- 13) See for example H.E. Rose, *A Course in Number Theory*, second edition, Oxford University Press 1994.
- 14) C.F. Gauss, *Disquisitiones Arithmeticae*, translated in English by A.A. Clarke, Yale University Press 1965. Gauss' work was published in Latin in 1801 and dedicated to the prince of Brunswick and Lunebourg. The first French translation was done as early as 1807 by Poulet-Delisle, probably at the instigation of Laplace : Ch.-Fr. Gauss, *Recherches arithmétiques*, traduites par A.-C.-M. Poulet-Delisle, à Paris, chez Courcier, Imprimeur-Libraire pour les Mathématiques, quai des Augustins 1807. Reproduced by the Editions Jacques Gabay, 92330 Sceaux 1989.
- 15) L.E. Dickson, *History of the Theory of Numbers*, vol. I : *Divisibility and Primality*, Dover 2005 ; unabridged republication of Publication Number 256 in Washington, D.C. by the Carnegie Institute of Washington in 1919.
- 16) D.A., part III .
- 17) This was already noticed by Gauss, D.A., article 62. According to Dickson, ref. 14 p. 184, this was also stated by J. Ivory in the Encyclopaedia Britannica of 1824.
- 18) Be careful that the absolute values of  $\alpha^v$  and  $-\alpha^v \pmod{p}$  are different so that they do not cancel when subtracted as in usual algebra.
- 19) D.A., part III, article 79.
- 20) This section A.2 essentially details and develops D.A.'s proposition 49.
- 21) For Gauss' approach in D.A., see article 84 and the other articles he indicates.
- 22) D.A., chap. III, articles 88-89.
- 23) G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Clarendon Press Oxford 1979, Chap. VII, 7.4 ; (first edition 1938).
- 24) See ref. 13 p.91. This is a special case of Gauss' proposition 90.
- 25) D.A., articles 86 and 90.
- 26) Gauss, D.A. chap. III, article 90, says, in Clarke's translation : *If some power of the number 2 higher than the second, e.g.  $2^n$ , is taken as modulus, the  $2^{n-2}$ th power of any odd number is congruent to unity* . Though this is perfectly correct, it is not equivalent to saying that the period is  $2^{N-2} \pmod{2^N}$  . As here shown the period is shorter if 2 is not a PR of  $\alpha$ , the base chosen, the congruence to unity occuring in this case before the index  $2^{N-2}$  . This is once more the illustration that in number theory, theory should constantly be confronted to numerical tests, so the more today that computational means are presently available that were not at Gauss' epoch.
- 27) F.J. Dyson and H. Falk, *Period of a Discrete Cat Mapping*, American Mathematical Monthly, 99, 603-614 (1992).
- 28) Number theory including Fibonacci series and their applications to various fields in physics and computer theory are elegantly developed in a no axiomatic way in the following book: M.R. Schroeder, *Number Theory in Science and Communication*, second edition, Springer-Verlag 1990.
- 29) J. Dayantis and J.-F. Paliarne, *A search of primes from lesser primes*, Journal of Discrete Mathematical Sciences and Cryptography, 10, 581-602 (2007).

## Polysemy of symbols: Signs of ambiguity

Ami Mamolo<sup>1</sup>

Department of Mathematics and Statistics  
Queen's University

**Abstract:** *This article explores instances of symbol polysemy within mathematics as it manifests in different areas within the mathematics register. In particular, it illustrates how even basic symbols, such as '+' and 'I', may carry with them meaning in 'new' contexts that is inconsistent with their use in 'familiar' contexts. This article illustrates that knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol.*

Keywords: mathematical registers; polysemy; symbols; transfinite arithmetic

---

<sup>1</sup> Queen's University  
Department of Mathematics and Statistics  
Kingston, ON  
K7L 3N6  
Email: [ami@mast.queensu.ca](mailto:ami@mast.queensu.ca)

Ambiguity in mathematics is recognized as “an essential characteristic of the conceptual development of the subject” (Byers, 2007, p.77) and as a feature which “opens the door to new ideas, new insights, deeper understanding” (p.78). Gray and Tall (1994) first alerted readers to the inherent ambiguity of symbols, such as  $5 + 4$ , which may be understood both as *processes* and *concepts*, which they termed *procepts*. They advocated for the importance of flexibly interpreting procepts, and suggested that “This ambiguous use of symbolism is at the root of powerful mathematical thinking” (Gray and Tall, 1994, p.125). A flexible interpretation of a symbol can go beyond process-concept duality to include other ambiguities relating to the diverse meanings of that symbol, which in turn may also be the source of powerful mathematical thinking and learning. This article considers cases of ambiguity connected to the context-dependent definitions of symbols, that is, the *polysemy* of symbols.

A polysemous word can be defined as a word which has two or more different, but related, meanings. For example, the English word ‘milk’ is polysemous, and its intended meaning can be determined by the context in which it is used. Mason, Kniseley, and Kendall (1979) observed that word polysemy in elementary school reading tasks was a source of difficulty – students demonstrated a tendency to identify the common meaning of words, despite being presented contexts in which an alternative meaning was relevant. Durkin and Shire (1991) discussed several instances of polysemous words within the mathematics classroom. They noted confusion in children’s understanding of expressions that had both mathematical and familiar ‘everyday’ meanings. In resonance with Mason, Kniseley, and Kendall (1979), Durkin and Shire found that “when children misidentified the meaning of an ambiguous word in a mathematical sentence, the sense they chose was often the everyday sense” (1991, p.75).

In addition to potential confusion between a word’s ‘everyday’ meaning and its specialized meaning within mathematics, learners are also often faced with polysemous terms within the mathematics register. Zazkis (1998) discussed two examples of polysemy in the mathematics register: the words ‘divisor’ and ‘quotient’. These words were problematic for a group of prospective teachers when confusion about their meanings could not be resolved by considering context – both meanings arose within the same context. In the case of ‘divisor’, attention to subtle changes in grammatical form was necessary to resolve the confusion. In the case of ‘quotient’, a conflict between familiar use and precise mathematical definition needed to be acknowledged and then resolved. Zazkis relates to the mathematics register Durkin and

Shire's (1991) suggestion that enriched learning may ensue from monitoring, confronting and 'exploiting to advantage' ambiguity.

I would like to continue the conversation on polysemy within the mathematics register, and extend its scope to consider the polysemy of mathematical symbols. This article examines the polysemy of the '+' symbol as it manifests in different areas within the mathematics register. The article begins with a reminder of the 'familiar' – addition and addends in the case of natural numbers – as well as a brief look at an example where meanings of symbols are extended within the sub-register of elementary school mathematics. Following that, I focus on two instances where meanings of familiar symbols are extended further: the first involves modular arithmetic, while the second involves transfinite arithmetic. I chose to focus on these cases for two reasons: (i) the extended meanings of symbols such as ' $a + b$ ' contribute to results that are inconsistent with the 'familiar', and (ii) they are items in pre-service teacher mathematics education.

This article presents an argument that suggests that the challenges learners face when dealing with polysemous terms (both within and outside mathematics) are also at hand when dealing with mathematical symbols by starting with 'obvious' and well-known illustrations of symbol polysemy in order to prepare the background to analogous but not-so-obvious observations. It focuses on cases where acknowledging the ambiguity in symbolism and explicitly identifying the precise, context-specific, meaning of that symbolism go hand-in-hand with understanding the ideas involved.

### **Building on the familiar: from natural to rational**

The main goal of this section is to establish some common ground with respect to 'familiar' meanings of symbols of addition and addends. In the subsequent sections, the meanings of these symbols will be extended in different ways, dependent on context. Their extensions will be explored so as to highlight ambiguity in meanings which can be problematic for learners should it go unacknowledged.

Since experiences with symbols in mathematics often start with the natural numbers, it seems fitting that this paper should start there as well. Natural numbers may be identified with cardinalities<sup>2</sup>, or 'sizes', of finite sets – where '1' is the symbol for the cardinality of a set with a single element, '2' the symbol for the cardinality of a set with two elements, and so on. With

---

<sup>2</sup> Natural numbers may also be identified with ordinals; however addition of ordinals is not commutative (Hrbacek and Jech, 1999), and thus in doing so one loses a fundamental property of natural number arithmetic.

such a definition, addition over the set of natural numbers may be defined as the operation which determines the cardinality of the union of two disjoint sets (Hrbacek and Jech, 1999; Levy, 1979). As noted earlier, a symbol such as ‘1+2’ can be considered a *procept*, and as such may be viewed as both the *process* of adding two numbers and also the *concept* of the sum of two numbers. For the purposes of this paper, it is enough to restrict attention to the concept of ‘1+2’ (and hereafter all other arithmetic expressions), though the process of ‘1+2’ is no less polysemous.

A more formal definition of addition over the set of natural numbers,  $\mathbb{N}$ , can be written as the following:

- if A and B are two disjoint sets with cardinalities  $a, b$  in  $\mathbb{N}$ , then the sum  $a + b$  is equal to the cardinality of the union set of A and B, that is, the set  $(A \cup B)$ .

Table 1 below summarizes the meanings of the symbols ‘1’, ‘2’, and ‘1+2’ when considered within the context of natural number addition:

Symbol	Meaning in context of natural numbers
1	Cardinality of a set containing a single element
2	Cardinality of a set containing exactly two elements
1+2	Cardinality of the union set

Table 1: Summary of familiar meaning in  $\mathbb{N}$

Sensitivity towards various meanings attributed to arithmetic symbols is endorsed by teacher preparation guides and texts, such as Van de Walle and Folk’s *Elementary and Middle School Mathematics*, which notes that “each of the [arithmetic] operations has many different meanings” and that “Care must be taken to help students see that the same symbol can have multiple meanings” (2005, p.116). Van de Walle and Folk highlight as an example the ‘minus sign’, which they observe has a broader meaning than ‘take away’. However, they seem to take for granted that their readers are familiar with exact mathematical meaning of arithmetic symbols. For instance, they introduce addition as a ‘big idea’ which “names the whole in terms of the parts” (p.115), but without explicitly defining addition over the natural numbers, nor distinguishing conceptually natural number addition from, say, rational number addition. Rather, they recommend that “the same ideas developed for operations with whole numbers should apply to operations with fractions. Operations with fractions should begin by applying these same ideas to fractional parts” (p.244). This advice has dubious implications both conceptually and

pedagogically when we consider the definition of natural (and whole) numbers as cardinalities of sets. Rational numbers do not have an analogous definition as cardinalities, and indeed, the idea that a set might contain  $\frac{1}{2}$  or  $\frac{1}{4}$  of an element is not meaningful. Instead, rational numbers may be described as numbers that can be represented as a ratio  $v : w$ , where  $v$  and  $w$  are integers.

Campbell (2006) warns against conflating whole number and rational number arithmetic, and suggests that merging the two ideas may be the root of both conceptual and procedural difficulties during an individual's transition from arithmetic to algebra. Campbell identifies a source for this confusion as the

“relatively recent development in the history of mathematics that has logically subsumed whole (and integer) numbers as a formal subset of rational (and real) numbers. This development appears to have motivated and encouraged some serious pedagogical mismatches between the historical, psychological, and formal development of mathematical understanding” (2006, p.34)

Campbell asserts that the set of natural (and whole) numbers are not a subset of the set of rational numbers, but rather are isomorphic to a subset of the rational numbers. As such, this distinction is significant as it carries with it separate definitions for the set of natural numbers (and its corresponding arithmetic operations) and the subset of the rational numbers to which it is isomorphic. In particular, although the symbols appear the same, their meaning in this new context is different, as illustrated in Table 2.

Symbol	Meaning in context of rational numbers
1	A ratio of integers equivalent to 1:1
2	A ratio of integers equivalent to 2:1
1+2	A ratio of integers equivalent to 3:1

Table 2: Summary of extended meaning in  $\mathbb{Q}$ 

Campbell suggests that although the

“standard view... is to claim that young children are simply not developed or experienced enough to grasp the various abstract distinctions and relations to be made between whole number and rational number arithmetic... it may be the case that the cognitive difficulties in children’s understanding of basic arithmetic is a result of selling short their cognitive abilities” (2006, p.34).

Thus, although it may seem cumbersome to distinguish between  $1 \in \mathbb{N}$  and  $1$  (or  $1.0$ )  $\in \mathbb{Q}$ , where  $\mathbb{Q}$  symbolizes the set of rational numbers, it is conceptually important. In a broad context, the operation of addition may be considered as a binary function, and as such, its definition depends on the domain to which it applies. Recalling Table 1, we may add another row:

Symbol	Meaning in context of natural numbers
1	Cardinality of a set containing a single element
2	Cardinality of a set containing exactly two elements
1+2	Cardinality of the union set
+	Binary operation over the set of natural numbers

Table 1B: Summary of familiar meaning in  $\mathbb{N}$ 

It is useful for purposes of clarity in this paper to distinguish between different definitions of the addition symbol as they apply to different domains. The symbol  $+_{\mathbb{N}}$  will be used to represent addition over the set of natural numbers,  $+_{\mathbb{Z}}$  to represent addition over the set of integers, and  $+_{\mathbb{Q}}$  to represent addition over the set of rational numbers.  $+_{\mathbb{N}}$  and  $+_{\mathbb{Q}}$  have, to apply Zazkis’s (1998) phrase, the ‘luxury of consistency’ – despite the different definitions,  $1 +_{\mathbb{N}} 2 = 3$  and  $1 +_{\mathbb{Q}} 2 = 3$ . However, if we consider summing non-integer rational numbers, there are pedagogical consequences for neglecting the distinction between natural number addition and rational number addition. In particular with respect to motivating and justifying the specific algorithms applicable to computations with fractions, and also with respect to interpreting student error. A

classic error such as  $\frac{1}{2} + \frac{1}{3} = \frac{2}{3}$  may be seen as a reasonable interpretation of Van de Walle and Folk's (2005) advice of applying whole number operations to fractional parts. Without distinction,  $\frac{1}{2} + \frac{1}{3}$  is, for a learner, equivalent to  $\frac{1}{2} +_N \frac{1}{3}$ . This latter expression is logically problematic: as a binary function,  $+_N$  is applicable only to elements in its domain – the set of natural numbers – in which the fractions  $\frac{1}{2}$  and  $\frac{1}{3}$  are not.  $\frac{1}{2} + \frac{1}{3} = \frac{2}{3}$  may be viewed as an algorithm that restricts the function  $+_N$  to elements of its domain (the two numerators, and the two denominators). Adequate knowledge of addition as an operation whose properties depend upon the domain to which it applies, offers teachers a powerful tool to address the inappropriateness of this improvised algorithm.

The following sections build on the idea of addition as a domain-dependent binary operation. They explore examples of two domains for which a 'luxury' of consistency is absent: (i) the set  $\{0, 1, 2\}$  and (ii) the class of (generalised) cardinal numbers. When clarification is necessary, the notation  $+_3$  will be used to represent addition over the set  $\{0, 1, 2\}$  (i.e. modular arithmetic with base 3), and  $+_\infty$  will be used to represent addition over the class of cardinal numbers (i.e. transfinite arithmetic). The sections take a close look at familiar and not-so-familiar examples of domains for which an understanding develops hand-in-hand with an understanding of the associated arithmetic operations.



### Extending the familiar: an example in modular arithmetic

Modular arithmetic is one of the threads of number theory that weaves its way through elementary school to university mathematics to teachers' professional development programs – it is introduced to children in 'clock arithmetic', it is fundamental to concepts in group theory, and it is a concept that has helped teachers develop both their mathematical and pedagogical content knowledge. This section considers the context of group theory. It takes as a generic example the group  $\mathbb{Z}_3$  – the group of elements  $\{0, 1, 2\}$  with the associated operation of addition modulo 3.

Within group theory the meanings of symbols such as 0, 1, 2, +, and 1+2 are extended from the familiar in several ways. As an element of  $\mathbb{Z}_3$ , the symbol 0 is short-hand notation for the congruence class of 0 modulo 3. That is, it is taken to mean the set consisting of all the integral multiples of 3:  $\{\dots -6, -3, 0, 3, 6, \dots\}$ . Similarly, the symbol 1 represents the congruence class of 1 modulo 3, which consists of the integers which differ from 1 by an integral multiple of 3, and 2 represents the congruence class of 2 modulo 3, which consists of the integers which differ from 2 by an integral multiple of 3. The symbol '+' also carries with it a new meaning in this context: it is defined as addition modulo 3. As Dummit and Foote (1999) caution:

“we shall frequently denote the elements of  $\mathbb{Z}/n\mathbb{Z}$  [or  $\mathbb{Z}_n$ ] simply by  $\{0, 1, \dots, n-1\}$  where addition and multiplication are *reduced mod* [modulo]  $n$ . It is important to remember, however, that the elements of  $\mathbb{Z}/n\mathbb{Z}$  are *not* integers, but rather collections of usual integers, and the arithmetic is quite different” (p.10, emphasis in original)

Pausing for a moment on the symbol '1+2', we might explore just how different the meaning of addition modulo 3 is from the 'usual integer' addition. Since the symbols '1' and '2' (in this context) represent the congruence classes  $\{\dots -5, -2, 1, 4, 7, \dots\}$  and  $\{\dots -4, -1, 2, 5, 8, \dots\}$ , respectively, the sum '1+2' must also be a congruence class. Dummit and Foote (1999) define the sum of congruence classes by outlining its computation. In the case of 1+2 (modulo 3), we may compute the sum by taking *any representative integer* in the set  $\{\dots -5, -2, 1, 4, 7, \dots\}$  and *any representative integer* in the set  $\{\dots -4, -1, 2, 5, 8, \dots\}$ , and summing them in the 'usual integer way' (i.e. with the operation  $+$  <sub>$\mathbb{Z}$</sub> ). Having completed this, the next step is to determine the final result: the congruence class containing the integral sum of the two representative integers. Defined in this way, addition modulo 3 does not depend on the choice of representatives taken

for ‘1’ and ‘2’. Thus, recalling the notation introduced in the previous section, sample computations to satisfy this definition include:

$$\begin{aligned} 1 +_3 2 &= (1 +_Z 2) \text{ modulo } 3 \\ &= (1 +_Z 5) \text{ modulo } 3 \\ &= (-2 +_Z -1) \text{ modulo } 3 \end{aligned}$$

all of which are equal to the congruence class 0.

Laden with new meaning, these symbols pose a challenge for students who must quickly adjust to a context where the complexity of such compact notation is taken for granted, and where inconsistencies arise between the symbols’ specialized meaning and their ‘familiar’, ‘usual’ meaning. Table 3 below summarizes the meanings of the symbols ‘1’, ‘2’, and ‘1+2’, and ‘+’ when considered within the context of  $\mathbb{Z}_3$ :

Symbol	Meaning in context of $\mathbb{Z}_3$
1	Congruence class of 1 modulo 3: $\{\dots -5, -2, 1, 4, 7, \dots\}$
2	Congruence class of 2 modulo 3: $\{\dots -4, -1, 2, 5, 8, \dots\}$
1+2	Congruence class of (1+2) modulo 3: $\{\dots, -3, 0, 3, \dots\}$
+	Binary operation over set $\{0, 1, 2\}$ ; addition modulo 3

Table 3: Summary of extended meaning in  $\mathbb{Z}_3$

The process of adding congruence classes by adding their representatives is a special case of the more general group theoretic construction of a quotient and quotient group – central ideas in algebra, and ones which have been acknowledged as problematic for learners (e.g. Asiala et al., 1997; Dubinsky et al., 1994). These concepts are challenging and abstract, and are made no less accessible by opaque symbolism. As in the case with words, the extended meaning of a symbol can be interpreted as a metaphoric use of the symbol, and thus may evoke prior knowledge or experience that is incompatible with the broadened use. In a related discussion of the challenges learners face when the meaning of a term is extended from everyday language to the mathematics register, Pimm (1987) notes that “the required mental shifts involved can be extreme, and are often accompanied by great distress, particularly if pupils are unaware that the difficulties they are experiencing are not an inherent problem with the idea itself” (p.107) but instead are a consequence of inappropriately carrying over meaning from one register to the other. A similar situation arises as learners must stretch and revise their understanding of a

symbol within the mathematics register – an important mental shift that is taken for granted when clarification of symbol polysemy remains tacit.

### **Beyond the familiar: an example in transfinite arithmetic**

Transfinite arithmetic may be thought of as an extension of natural number arithmetic – its addends (transfinite numbers) represent cardinalities of finite or *infinite* sets. Transfinite arithmetic poses many challenges for learners, not the least of which involves appreciating the idea of ‘infinity’ in terms of cardinalities of sets. Before one may talk meaningfully about polysemy and ambiguity in transfinite arithmetic, it is important to first develop some ideas about ‘infinity as cardinality’, which is where this section will begin.

Infinity is an example of a term which is polysemous both across and within registers. The familiar association of infinity with endlessness is extended into the mathematics register in areas such as calculus where the idea of *potential infinity* is indispensable. Potential infinity may be thought of as an inexhaustible process – one for which each step is finite, but which continues indefinitely. In calculus for example, the idea of limits which ‘tend to’ infinity relates the notion of an on-going process that is never completed. This extension across registers preserves some of the meaning connected to the colloquial use of the term ‘infinity’, however it is distinct from intuitions which, say, connect infinity to endless time or to the all-encompassing (see Mamolo and Zazkis, 2008). Within the mathematics register, the term ‘infinity’ is extended further to the idea of *actual infinity*, which is prevalent in the field of set theory. Actual infinity is thought of as a completed and existing entity, one that encompasses the potentially infinite. The set of natural numbers is an example of an actually infinite entity – it contains infinitely many elements and, as a set, exists despite the impossibility of enumerating all of its elements. The cardinality of the set of natural numbers is another instance of actual infinity; it is also the smallest transfinite number.

Transfinite numbers are generalised natural numbers which describe the cardinalities of infinite sets. As implied, infinite sets may be of different cardinality: the set of natural numbers, for example, has a different cardinality than the set of real numbers, though both contain infinitely many elements. Cardinalities of two infinite sets are compared by the existence or non-existence of a one-to-one correspondence between the sets. Two sets share the same cardinality if and only if every element in the first set may be ‘coupled’ with exactly one element in the second set, and vice versa. This is a useful approach, and I will return to it when illustrating properties of transfinite arithmetic. The point I am trying to make here is that the concept of a

transfinite number, which intuitively may be thought of as an ‘infinite number’, requires extending beyond the familiar idea of infinity as endless (and thus unsurpassable). Also, in resonance with Pimm’s (1987) observation regarding negative and complex numbers, the concept of a transfinite number “involves a metaphoric broadening of the notion of number itself” (p.107). In this case, the broadening includes accommodating some arithmetic properties which are both unfamiliar and unintuitive.

As in the case with arithmetic over the set of natural numbers, transfinite arithmetic involves determining the cardinality of the union of two disjoint sets. The crucial distinction is of course that at least one of these sets must have infinite magnitude – its cardinality must be equal to a transfinite number. To illustrate some of the distinctive properties of transfinite arithmetic consider, without loss of generality, the cardinality of the set of natural numbers, denoted by the symbol  $\aleph_0$ . Imagine adding to the set of natural numbers,  $\mathbb{N}$ , a new element, say  $\beta$ . This union set  $\mathbb{N} \cup \{\beta\}$  has cardinality equal to  $\aleph_0 + 1$  – there is nothing new here. However, each element in  $\mathbb{N}$  can be ‘coupled’ with exactly one element in  $\mathbb{N} \cup \{\beta\}$ , and vice versa. By definition, two infinite sets have the same cardinality if and only if they may be put in one-to-one correspondence, thus the cardinality of  $\mathbb{N}$  is equal to the cardinality of  $\mathbb{N} \cup \{\beta\}$ . As such,  $\aleph_0 = \aleph_0 + 1$ . Similarly, it is possible to add an arbitrary natural number of elements to the set of natural numbers and not increase its cardinality, that is  $\aleph_0 = \aleph_0 + v$ , for any  $v \in \mathbb{N}$ , and further  $\aleph_0 + \aleph_0 = \aleph_0$ .

This ‘tutorial’ in transfinite arithmetic is relevant to the discussion on polysemy as it illustrates how the symbol ‘+’ in this context is quite distinct in meaning from addition over the set of natural numbers. Whereas with ‘+<sub>N</sub>’ adding two numbers always results in a new (distinct) number, with ‘+<sub>∞</sub>’ there exist non-unique sums. Further, since the concept of a set of numbers must be extended to the more general ‘class’ of transfinite numbers, the symbol ‘1’ in the expression ‘ $\aleph_0 + 1$ ’ also takes on a slightly new meaning since it must be considered more generally as a class (rather than set) element<sup>3</sup>. Extended meanings connected to transfinite arithmetic are summarized in Table 4:

Symbol	Meaning in context of transfinite arithmetic
1	Cardinality of the set with a single element; class element
$\aleph_0$	Cardinality of $\mathbb{N}$ ; transfinite number; ‘infinity’

<sup>3</sup> For distinction between set and class, see Levy (1979).

$\aleph_0 + 1$	Cardinality of the set $\mathbb{N} \cup \beta$ ; equal to $\aleph_0$
$+$	Binary operation over the class of transfinite numbers

Table 4: Summary of extended meaning in transfinite arithmetic

A specific challenge related to the polysemy of  $+$  in this context derives from the existence of non-unique sums, a consequence of which is indeterminate differences. Explicitly, since  $\aleph_0 = \aleph_0 + v$ , for any  $v \in \mathbb{N}$ , then  $\aleph_0 - \aleph_0$  has no unique resolution. As such, the familiar experience that ‘anything minus itself is zero’ does not extend to transfinite subtraction. This property is in fact part and parcel to the concept of transfinite numbers. Identifying precisely the context-specific meaning of these symbols ( $‘+_{\infty}’$  and  $‘-_{\infty}’$ ) can help solidify the concept of transfinite numbers, while also deflecting naïve conceptions of infinity as simply a ‘big unknown number’ by emphasizing that transfinite numbers are different from ‘big numbers’ since they have different properties and are operated upon (arithmetically) in different ways.

In this section, to address issues of polysemy of symbols, it was necessary to first glance at the polysemy of the term infinity. It is a complex concept that can encompass different connotations across and within different registers. Within mathematics, it is difficult to think of infinity – even in the context of transfinite numbers – without imagining that well-known symbol  $‘\infty’$ . Informally, the symbol  $‘\aleph_0 + 1’$  might be thought of as  $‘\infty + 1’$ . This informal symbolism suggests the idea of adding 1 to a ‘concept’ rather than a ‘set number’, of adding 1 to endlessness. Notwithstanding the formal use of  $‘\aleph_0’$ , an intuition of  $‘\infty’$  may persist (if only tacitly), carrying with it all sorts of inappropriate associations.

### **Concluding Remarks**

This article examined instances of symbol polysemy within mathematics. The intent was to illustrate how even basic symbols, such as ‘+’ and ‘1’, may carry with them meaning that is inconsistent with their use in ‘familiar’ contexts. It focused on cases where acknowledging the ambiguity in symbolism and explicitly identifying the precise (extended) meaning of that symbolism go hand-in-hand with developing an understanding of the ideas involved. While this article focused on particular examples of distinguishing among the symbolic notation for arithmetic over the set of natural numbers, rational numbers, equivalence classes, and transfinite cardinals is fundamental to appreciating the subtle (and not-so-subtle) differences among the elements of those sets, this argument has broader application. I suggest that the challenges learners face when dealing with polysemous terms (both within and outside mathematics) are also at hand when dealing with polysemous symbols. Just as knowledge of languages such as English include “learning a meaning of a word, learning more than one meaning, and learning how to choose the contextually supported meaning” (Mason et al., 1979, p.64), knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol. Further, echoing Pimm’s (1987) advice and extending its scope to include mathematical symbols:

“If ... certain conceptual extensions in mathematics [are] not made abundantly clear to pupils, then specific meanings and observations about the original setting, whether intuitive or consciously formulated, will be carried over to the new setting where they are often inappropriate or incorrect” (p.107).

Sfard (2001) suggests that symbols – such as the ones discussed here, but also in a more general sense – are not “mere auxiliary means that come to provide expression to pre-existing, pre-formed thought” but rather are “part and parcel of the act of communication and thus of cognition” (p.29). As such, attending to the polysemy of symbols, either as a learner, for a learner, or as a researcher, may expose confusion or inappropriate associations that could otherwise go unresolved. Research in literacy suggests that students “may rely on context when a word does not have a strong primary meaning to them but will choose a common meaning, violating the context, when they know one meaning very well” (Mason et al., 1979, p.63).

Further research in mathematics education is needed to establish to what degree analogous observations apply as students begin to learn ‘+’ in new contexts.

### Acknowledgements

I would like to thank Rina Zazkis for her insightful comments and encouragement on an early draft of this paper.

### References

- Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S. and Oktaç, A. (1997). Development of students’ understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior*, 16(3), 241–309.
- Byers, W. (2007). *How Mathematicians Think: Using Ambiguity, Contradiction and Paradox to Create Mathematics*. Princeton, NJ: Princeton University Press.
- Campbell, S. (2006). Understanding elementary number theory in relation to arithmetic and algebra. In Zazkis, R. and Campbell, S. (eds), *Number Theory in Mathematics Education: Perspectives and Prospects*. Mahwah, NJ: Lawrence Erlbaum Associates, pp. 19-40.
- Dubinsky, E., Dautermann, J., Leron, U., and Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27, 267-305.
- Dummit, D. S. and Foote, R. M. (1999). *Abstract Algebra*. New York, NY: John Wiley & Sons Inc.
- Durkin, K. and Shire, B. (1991). Lexical ambiguity in mathematical contexts. In Durkin, K. and Shire, B. (eds), *Language in Mathematical Education: Research and Practice*. Milton Keynes, Bucks: Open University Press, pp.71-84.
- Gray, E. and Tall, D. (1994). Duality, ambiguity and flexibility: A ‘proceptual’ view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140.
- Hrbacek, K. and Jech, T. (1999). *Introduction to set theory*. New York, NY: Marcel Dekker, Inc.
- Levy, A. (1979) *Basic Set Theory*, Berlin, New York, Springer-Verlag.
- Mamolo, A. and Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education*, 10(2), 167 – 182.
- Mason, J.M., Kniseley, E., and Kendall, J. (1979). Effects of polysemous words on sentence comprehension. *Reading Research Quarterly*, 15(1), 49-65.

- Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classroom*, London, Routledge.
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communication to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13 – 57.
- Van de Walle, J.A. and Folk, F. (2005) *Elementary and Middle School Mathematics: Teaching Developmentally*, Toronto, ON, Pearson Education Canada
- Zazkis, R. (1998) ‘Divisors and quotients: Acknowledging polysemy’, *For the Learning of Mathematics*, 18(3), 27-30.





## **Cultural conflicts in mathematics classrooms and resolution – The case of immigrants from the Former Soviet Union and Israeli “Old timers”**

*Miriam Amit<sup>j</sup>*

*Ben-Gurion University, Israel*

*Abstract: This paper describes a singular process that has been transforming mathematics education in Israel over the past 20 years, as a result of a massive influx of mathematics teachers from the former Soviet Union (FSU). It traces the key points of conflict that marked the initial contact between Israel's mathematical and educational culture and the codes and values brought with the immigrant teachers from the FSU. It then shows how this conflict is gradually becoming resolved, as the two disparate cultures merge into a single, new culture, based on 'the best of both worlds.' This case, we claim, can serve as an example of the importance - and the benefit - of relations of mutual influence and stimulation between different groups in today's climate of migrating peoples and mixing cultures.*

**Keywords:** cultural conflict; cultural dimensions of mathematics teaching; educational culture; former Soviet Union; immigrant teachers; mathematics teachers; Israel; Israeli educational system; teacher identity; values

### **Socio-historical Background**

"Why Learn Math? Because it Develops Orderly Minds!"

This quote is attributed to the famous eighteenth century Russian mathematician, scientist, poet, and linguistics reformer Mikhail Lomonosov (1711-1765). Lena, who immigrated to Israel in 1995, remembers seeing the slogan hanging in her math class in Ukraine. She admits wistfully, yet proudly, that she misses order, clarity, and a sense of belonging. For Lena and thousands of other FSU immigrants teaching math in Israel, Lomonosov's inspiring message resounds far beyond mathematics. The importance of "the sense of order" in life that math provides is part of an ongoing cultural legacy (Amit & Burde, 2005).

Usually, when we speak of cultural gaps in education, we refer to gaps that exist between teachers and their students, or between students of different backgrounds (Sfard & Prusak, 2005). This study, however, addresses a cultural divide that was formed within the teaching community, when a wave of mass-immigration to Israel from the former Soviet Union brought together two groups of teachers from very different backgrounds<sup>ii</sup>. Over the past two decades, more than 900,000 immigrants from the FSU, many of them educated in science or engineering, have arrived in Israel and made their way into the job market, particularly as teachers of mathematics and physics (Darr & Rotschild, 2004). Though this is, in fact, the second wave of Soviet immigration, following an earlier one in the 1970s, its impact on Israeli society was far

more significant, since it was both quantitatively much larger (constituting a 15% addition to the country's population of six million), and qualitatively less prone to assimilation. (This latter point is tangible in the two waves' respective treatment of names. While in the first "Igors" tended to change into more Israeli sounding "Igal," children born in Israel in the '90s were still given "Russian" names like "Natalia" or "Oleg").

The immigrants arriving from the FSU were marked by a shared perception of the immense value and importance of higher education. For them, such education was both a duty and an aspiration, particularly in the fields of natural science, engineering and mathematics.

Many of the immigrants who became teachers in Israel had been teachers in the FSU, while others had been engineers or scientists - products of the one level or another of the Soviet educational system. Though the hometowns of these people were widely dispersed throughout the Soviet Union, their concept of mathematics education was strikingly homogenous, a consistency most probably born of the Soviet educational system's tight centralization. The new immigrant teachers were set to work side by side with the veteran teachers already present in the educational system, teaching classes composed of a mix of immigrant and native-born students, in which the sole language of instruction was Hebrew. While the need to teach in a foreign language was originally assumed to be a potential source of great difficulty, it proved in retrospect to be far less problematic than the foreignness the immigrants discovered in the culture and mentality underlying the Israeli approach to mathematics education. The encounter between the FSU immigrant math teachers and their Israeli counterparts, and the conflicts it produced, led to changes in the country's math education culture. The uniqueness of these changes was that they occurred 'from the bottom up,' originating in the teachers and their teaching practices rather than in politicians, market forces, industrialists, or researchers (Amit & Fried, 2002).

The experiences of the immigrant mathematics teachers, and how these contrasted with their expectations, are suggestive of the importance of acknowledging that even something as seemingly 'objective' as mathematics is influenced by cultural concerns. This notion that an inseparable link exists between math education on the one hand and values, culture, and society, on the other has long been acknowledged by the mathematics education research community (Bishop, 1988; Amit, 2000; Seah 2004; Sfard & Prusak, 2005). It has not yet, however, been similarly acknowledged by those engaged in the practice of mathematics education (such as teachers and policy makers), which may account for why these elements were not taken into account as factors in the immigrant teachers' professional integration.

The entering teachers were provided with a framework of preparation and assimilation into the Israeli educational system, but despite the expectation that mathematics could be universally taught anywhere, all parties were surprised to discover that this was not the case. The integration programs focused solely on acquainting the newcomers with the Israeli syllabus, and other logistical issues pertaining to the management of Israeli schools. They included no element of cultural or social acclimatization. Tamara, for instance, came to Israel with a doctorate degree in mathematics and twenty-two years of teaching experience, and had to take a year-long preparatory course before receiving an Israeli teaching certificateiii.

Tamara: At first I felt that the course had nothing to offer me . . . I was [sure it] would be a waste of time, a mere formality, and that I'd sit there for a year, and start working the following year. I was certain that I'd be able to step into a classroom and just begin teaching math. [I believed

that] although people's [cultural] natures differ, traditional mathematics is the same everywhere, and it was a waste of time to go to an absorption centre– Was I ever wrong! [emphatically stated].

### **Research Aims and Scope**

This paper aims to illuminate some of the cultural differences that emerged between the two groups of mathematics teachers working in Israel, and to show how these are gradually bringing forth a new mathematics education culture that draws on elements from both sides. The research on which it is based has been carried out over several years. The bulk of the data on FSU educational culture was gathered over the course of three intensive years of personal interviews and discussions, in groups of 3-5, with 14 teachers from the FSU. The confrontations with 'native' Israeli culture were observed both within such discussions, and in the hundreds of lessons, taught by immigrant and non-immigrant mathematics teachers, that the author of this paper audited during years spent working for the ministry of education.

All of the immigrant teachers participating in the study had held the equivalent of a Masters degree, and had teaching experience ranging from 18 to 30 years, the last of them spent teaching in Israel. All had gone through the preparatory course in Israel, had received a teaching certificate, and had good command of Hebrew. Clearly, this short paper cannot provide an exhaustive account of the results of over ten years of research, and the short summary provided here is just the 'tip of the iceberg.'

### **From Conflict to Collaboration- A Cultural Evolution**

The perceptions of mathematics education held by the immigrant and native Israeli teachers were separated by a cultural chasm – one that initially seemed insurmountable, producing great fissures in Israel's mathematics education system. These cultural gaps, however, are gradually being bridged. Though this process began with a meeting – and a confrontation - between two distinctly separate cultures, in time, the rigid borderline between the cultures began to crack, and they started to slowly leak into one another, progressing toward a final stage in which they merge into a new, composite culture that utilizes elements of both.

### **Conflicts in Values, Method and Practice: Primary Sources of Friction**

*Importance and function of mathematics education:* In Israel, as in many other 'Western' cultures, the study of mathematics is generally perceived as a means to an end; something one must study in order to qualify for training in certain privileged fields (such as medicine, computers, engineering). As a result, for those aspiring to succeed in other directions (achieving success in business, for instance) mathematics are a low priority – a point reflected in the recent dramatic drop in students specializing in mathematics. In the culture from which the immigrant teachers arrived, on the other hand, the study of mathematics was far more of an end unto itself, a field implicitly associated with high status and prestige.

Tamara: Education was considered a goal in its own right, as well as a key to a better way of life, a higher social position, and a larger income. This was the Soviet view. In Israel there are other paths to success, such as opening a business. In the FSU math was the leading subject in education, and as such was preeminently regarded.

*Teaching methods and syllabus:* While in both Israel and the FSU mathematics education is based upon a central, government imposed syllabus, the degree of centralization in the latter system is greater by far. In Israel, the path by which the government-decided end result is reached is left to the discretion of individual schools and teachers, allowing for the use of an ever-shifting range of teaching styles and textbooks. In the FSU, on the other hand, the government provides a single approved teaching approach and set of textbooks, allowing, at the expense of the relative freedom inherent in the Israeli system, a far greater educational stability over both time and space. The stark contrast between the approaches of these two cultures is evident in the definition of what constitutes "freedom" that arises from this statement by the USSR Academy of Pedagogical Sciences - the official body for research and applied education in the Soviet Union. The statement declares that "research on the new methods . . . frees teachers from repeated trial and error, from discovering what science already knows, and applying ideas that are unrealistic for the ordinary school" (Dunston & Suddaby, 1992, P. 9). Where an Israeli 'old-timer' might perceive the rigid Soviet educational system as limiting teachers' freedom by restricting their independence and creativity, this same system may be perceived from another direction as providing freedom from time-consuming responsibility and potentially wasted effort.

Boris describes the difference between Israeli and Soviet teaching methods in the discussion below:

Boris: There we had many didactic materials, booklets telling us how it [the teaching] should be, how it [the curriculum] should be constructed, how to teach...

Researcher: Who publishes these booklets?

Boris: Usually these were pretty formal things...there was a lot of monitoring there of the methods of teaching and working, and you could get a very unpleasant warning if you didn't follow the general spirit.

Researcher: And the 'general spirit' was decided in Moscow?

Boris: More or less.

Researcher: Didactically, do you have more freedom in Israel?

Boris: Yes, decidedly so. Here they do not monitor the didactic methods a teacher uses in his work. He can basically use any method he wants to succeed, they measure by the success of the end result and not by the way you get there.

Researcher: And you?

Boris: It may sound strange, but I felt very comfortable almost from the beginning in Israel with the option of choosing what's comfortable for me. It fits my character.

Upon arriving in Israel, immigrant teachers felt the lack of the ‘top down’ approach to which they had been accustomed, wanting precise information about what should be taught when and how, and which book should be used in doing so. Even as they learned to appreciate the increased freedom, they nevertheless lamented the loss of the FSU system’s predictability and long-term support.

The two cultures also differed in the larger amount of time and resources allotted to mathematics education in the FSU – where even far-flung schools were equipped with special ‘math-rooms,’ where mathematics was taught only in the morning, and where the smaller selection of other available study-topics meant that 8 classes could be devoted to mathematics every week. This last point is reflective of a basic difference in the two cultures’ approach to learning, whereby the Israeli system seeks to give students a smaller ‘taste’ of a large assortment of topics, while in the FSU a select group of priority topics are taught in greater depth - a difference that also shows in each culture’s choice of mathematical topics. Teachers from the FSU, who had been used to a curriculum highly focused on arithmetic, algebra and Euclidean geometry, were violently opposed to the Israeli reduction of these topics in favour of such additions as calculus, vectors, and (particularly) statistics. The lack of connection in Israeli schools between mathematics and other sciences, especially physics, was also difficult for the immigrant teachers, who had been schooled in multiple sciences in tandem, and in their connection to one another. They found the mathematics taught in Israel to be ‘stripped down’ by its disconnection from its sister-sciences.

Pedagogy and classroom management: Another source of friction and anxiety were the pedagogical differences between the two cultures. Mathematics education in the FSU was based largely on a method of ‘drill and practice,’ a ‘hard work’ approach based on the solving of many exercises and the development of technical skills. An FSU university entrance exam, for instance, might require a student to simplify the following complex algebraic expression:

$$\left(\frac{a}{3(a^2 + 1)^{1/2}} - (2a^2 + 1 + a\sqrt{4a^2 + 3})^{1/2} \cdot (2a^2 + 3 + a\sqrt{4a^2 + 3})^{-1/2}\right)^2$$

Such a show of algebraic technical skill would not be required of a high school graduate in Israel, which has opted, like much of the western world, to relinquish technique (making use of calculators etc), and stress instead an ‘overall comprehension’ of the topic, which does not necessarily centre on correct answers to individual exercises.

These differences go beyond the debate over pedagogical approaches and emphases, and are strongly rooted in both the values and the pragmatic reality of the Former Soviet Union. On an ethical level, hard work in itself was a virtue in the Soviet Union.

“Soviets hold that complex calculations inculcate good habits of hard work, while British (and others in the West) feel that mathematics is a subject [that teaches that] by hard thinking it is possible to avoid hard work” (Muckle, 1988, p. 58).

Furthermore, Algebraic skills demand very little independent creative thinking. This was perfectly acceptable since creative thinking was a skill the Soviet regime did not want to develop. At the pragmatic level, immigrant teachers have an ingrained belief, probably imbued in them from the pragmatism of the Soviet regime, that the solution of a problem is correct only if it arrives at a correct product. The Israeli system, on the other hand, holds the constructivist view that problem solution is a process, and that a student ought to be given credit for a partial answer, even if the final product is missing or incorrect. One FSU immigrant teacher told us that the algebraic approach of prioritizing process over product meant that: “The [students] are like engineers who know how to design things but not how to build them” (Tamara) – i.e. that they understand theoretically how a problem is to be solved, but are not necessarily capable of actually solving it.

The Israeli classroom environment proved to be a further source of contention, as it is far more lax than that of the FSU, which was based on rigid discipline and the absolute authority of the teacher. In Israel, students might challenge a teacher’s authority by speaking in class, expressing reluctance to complete ‘too much’ homework, or demanding further explanation if something is not understood. This, combined with the need to work in a new, unfamiliar language, could often be a blow to the immigrant teachers’ self-confidence.

Yafa, an Israeli ‘old-timer’ who works closely with immigrant colleagues, told us:

Yafa: The most prominent thing about [immigrant teachers] is their emphasis on technique and hard work. They pay attention to doing exercises, homework and student effort...and they have a certain intolerance when it comes to student behaviour. Students must behave respectfully and not be rude to the teacher...speak at the proper time and not interrupt. In their opinion – a student who knows how to behave is a good student.

Researcher: Behave according to their norms?

Yafa: Right, yes, of course. I think it’s not that different from the expectations of the veteran Israeli teachers, but because we have also spent some time living in this place (said emphatically) and we’re familiar with this behaviour, we are a bit more tolerant...of this behaviour, though we don’t agree with it.

Regarding differences in teaching method, Yafa adds:

“We try to vary things with games and projects, they emphasize technique and repetition. As my friend told me, “students need to know first to work and then to play.”

The immigrant teachers' struggle to adjust to Israel's constructivist views on education is clear from statements such as:

“There we knew how to teach properly, and we succeeded at what we were doing. We knew what was best for [the students] . . . Students cannot discover math rules on their own . . . Why should they? We can teach them the correct ones . . . The students’ role is to learn in any situation; they must practice constantly, do heavy loads of homework, and be very organized (especially in their notebooks), then their minds will be orderly . . . Why does a student have to

look for a mathematical explanation . . . what is wrong with receiving a good explanation?”  
(Lena)

The primary goal of teaching mathematics: A final pedagogical difference arose from the deep divergence of values that separated the two cultures’ conceptions of the purpose of teaching mathematics in school and what the primary goal of mathematics education should be. While the Israeli educational system aspired to the promotion of equality and to achieving an evenness in the academic level of all the pupils in a class, teachers arriving from the FSU saw mathematics classes as a site for identifying and promoting only those among their students who were mathematically gifted. This opposition led to conflicting views regarding the proper allocation of time and resources amongst students; where, for instance, should the best teachers be sent? To aid those who are weak in catching up, or to guide those already strong to greater heights? Here, the immigrant teachers’ desire to advance their students’ excellence conflicted with the budgetary constraints of a society primarily preoccupied with finding ways of providing equal educational opportunities for its socio-economically struggling communities.

Equality [was] top priority. No need to invest in the better students because they will manage on their own. And anyway, in practice there were no resources for it because we were responsible firstly for the weak and mediocre students (Yafa).

Yakov, another ‘old-timer,’ corroborates this point, telling us: “Equality and excellence are not mutually exclusive, but sometimes we get carried away, worrying so much about the equality that we miss the excellence.”

Evaluation and confidentiality: The two cultures differed greatly in their views of how evaluation is to be carried out and how information is to be shared between all of the parties involved in the educational process (i.e. students, teachers, parents, supervisors). In Israeli schools, for instance, preserving the confidentiality of a student's status is considered paramount. The teacher is therefore not allowed to make any potentially humiliating or hurtful public references to a student, or to divulge the names of particularly strong or weak students in an all-parent forum. Accordingly, Israeli schools do not employ the FSU method of in-class evaluation, wherein every session contains an element of testing in which pupils come to the board and receive a grade for their performance.

“Every lesson I had to give the pupils a grade, in other words, I had to test them. If the vice-principal, principal, or supervisor examined the teacher’s grade book (that was left at school for supervision every day), and it wasn’t in order, the teacher would be in deep trouble with the authorities. (Maria)



While both cultures made use of national exams, in the FSU these were given at the end of every school year and included an oral examination. In Israel in the 1990s, on the other hand, mathematics exams were given only at the end of high-school (this changed in the year 2000), and contained no oral elements whatsoever. In a further step towards relaxing its demands on the student, Israeli mathematics education divides them into three level groups at the age of 15-16, and the difficulty of their final matriculation is adjusted to their skill level (Amit & Fried, 2002). Israeli exams also nearly always contain an element of choice, so that students may theoretically still get full marks in an exam containing a question they do not know how to answer. The new immigrant teachers define this approach as “a recipe for laziness,” since the FSU, in contrast, maintains a single level of study for all students (except a gifted minority that receive extra instruction), and introduces no element of choice, though the level of the questions in the national exam rises as the exam progresses.

This aspect of cultural divergence extends beyond the evaluation of students, and is marked by differences in the evaluation of teachers as well. In Israel, teachers were expected to make a periodic report to their supervisors, and these supervisors (i.e. principles, or inspectors) attended lessons rarely, and only after coordinating their visit with the teacher in advance. In the FSU, on the other hand, teachers submitted weekly written reports to their supervisors detailing the progress of every lesson. Supervisors of different levels also entered teachers' lessons several times a year, without announcing their arrival in advance.

Finally, the manner and frequency in which information was to be shared with students' parents was also a point of contention. In Israel, parents typically receive written reports from their children's homeroom teachers three to four times a year and meet with them on an individual basis at least twice. Meetings with the teachers of particular subjects (such as mathematics) are optional. In the FSU, however, parents receive a weekly report of their child's grades and of the activities in class, a report that they must verify having received with their signature. Teachers there do not commonly meet parents individually, but general parent-teacher conferences are held three to four times a year in which the teacher reports on the class's progress, commonly singling out excellent pupils, and sometimes censuring problematic ones by name before all the parents. Furthermore, homeroom teachers in the FSU are required to visit their pupil's homes, and, in the event that any misbehaviour on the students' part goes unacknowledged by their parents, the school was authorized to refer the matter to the parents' place of work, so that the students' behaviour can potentially jeopardise the professional position and salary of their parents. (This latter system was in practice during the communist regime, and remained intact, due to inertia, in the first years of perestroika.)

### **Merging Cultures – A Mutual Exchange of Values Breeds a New Educational Approach**

The teachers immigrating from the FSU left a society based on hierarchy, uniformity and discipline, coming instead to one that – for good or ill - promotes equality, diversity, choice, and autonomy. The FSU's centralized educational system supports the teacher's honored status, individual excellence, and competitiveness, and channels enormous resources to the

advancement of education. In contrast, the immigrant teachers entered an educational system that grants teachers and pupils a generous portion of autonomy, encourages diversity in teaching methods, and allocates only frugal funds to math education. Though the conflicts generated by these many differences led a minority of immigrant teachers to break away from the mainstream and found parallel systems based on FSU values, most were ultimately absorbed into Israel's central educational framework.

While this absorption in itself is not, perhaps, out of the ordinary in immigrant situations, what stands out about this particular case is that the process of change and adjustment was not unidirectional, but occurred on the 'native' side as well. Even as the FSU immigrants worked to adopt the cultural codes that governed the Israeli educational system, that system was being impacted and reformed by its contact with the new culture the immigrants had brought with them from their former homeland.

One reason for the occurrence of such a mutual degree of influence is the sheer numbers of FSU immigrants that flowed into the Israeli education system, which meant that, though 'technically' a minority, their numbers were soon rivalling those of their native Israeli counterparts. Secondly, these teachers arrived bearing the FSU's 'aura of success' in mathematics, a reputation that granted them a fairly high standing in their field, despite their status as immigrants and newcomers. Under such dual conditions of sharp differences coupled with an unusual measure of equality, these two groups were charged with the task of coexisting under a single educational roof – a circumstance that ultimately led to a mutual cultural 'leakage.'

From this 'leakage' there emerged a new model of mathematics education, one that borrowed and combined elements from each of the cultures on which it was based:

Pedagogical adjustments – combining teacher-student reciprocity with technical expertise. In a synergetic fusing of the two pedagogical approaches, the Soviet virtues of hard work, repetition, practice and technical skill are now being acknowledged, while the FSU teachers have begun to adopt a more open 'bottom up' approach in their relation to their students.

Syllabus and teaching style – finding the balance between liberty and consistency. The immigrant teachers have learnt to appreciate the benefits of the greater freedom the looser Israeli curricular restraints allow them in choosing their own teaching styles and textbooks. The Israeli teachers, in turn, have accepted the value of the rigorousness and order espoused by the Soviet system in providing a more stable and coherent teaching and learning experience.

Stepping up student evaluation. The evaluation of pupils' achievement in Israel has become much more centralized and institutionalized, under the auspices of the Ministry of Education. It is worth noting that this change was not initiated by the immigrant teachers, being the result, rather, of concerns over Israel's ranking in international tests. Their strong presence was, however, a factor in the readiness with which these striking changes were adopted, since they were already accustomed to and in favor of the use of systematic external evaluation.

Merging values - joining equality with excellence. While the Israeli educational system continues to strive for equality in its pupils, it has also adopted the Soviet awareness of the value

of recognizing and promoting excellence. This new legitimization of aspiring to excellence has led to the rise of special programs for students who are particularly talented in mathematics, many of which are led by experienced teachers from the FSU. While some of these programs were set up within the schools themselves (such as special classes for 'math speakers'), excellence has remained a comparatively low state priority in terms of resource allocation. Most programs for mathematical excellence have therefore taken the form of extracurricular 'mathematics clubs.'

### **The Kidumatica Math Club**

"Kidumatica," a club "dedicated to advancing math excellence in the Negev," is one of the finest examples of the Israeli initiatives for excellence that arose under the influence of FSU cultural values. The club was founded by the author of this paper at the Ben-Gurion University of the Negev, and is designed for the advancement and development of young students (aged 10-17) of high cognitive potential, many of whom come from lower socio-economic strata. The club's main aim is to enhance mathematics-based reasoning, logic skills, scientific orientation, and creative and multi-directional thinking in solving unconventional problems. Most of Kidumatica's teachers are immigrant mathematicians, who derive immense satisfaction from this work, saying that it is "like breathing the oxygen-rich air from 'back there'." The founder of the club is herself an Israeli 'old-timer,' who sought to combine in Kidumatica the values of excellence acquired from her many years of contact with FSU teachers with the fundamental strive for equity on which she was raised. On the one hand, therefore, the club addresses Israel's social need to provide equal opportunity to diverse populations; on the other, it promotes mathematical excellence in the Soviet spirit. Kidumatica's successful pooling of key elements from both the Israeli and Soviet mathematics education cultures has proven that ideological fusion or integration is possible.

### **Prospects**

In this article we traced a process that began with a friction in the values and expectations of two societies, each comfortable with - and confident in the virtues of - their own way. The veteran Israelis entered the breach with the force of inertia and maintaining the status-quo, a force countered on the immigrants' side by their unusual size as a minority and the halo of mathematical success surrounding the country of their birth. However, these two different mathematical cultures have been gradually 'leaking into one another' and reforming into a new approach to mathematics education that incorporates elements of both.

The merging of these two cultures and the ironing out of their differences is by no means a finished process, and not all of the cultural conflicts have as yet been resolved. The issue of class discipline, for instance, continues to be particularly troublesome to the ex-Soviet teachers, as are such Israeli curricular choices as the limitation of the study of geometry and the insistence on teaching statistics. However, mutual respect and recognition are gradually breeding a model that takes the best of both worlds.

While this case is rare in the sheer enormity of the population change incurred by the FSU immigration, it nevertheless has applicable lessons to teach in today's world of social and cultural mobility, where the population of many countries is being infused with varying quantities of immigrants. Firstly, our study highlights the presence of cultural conflict as a pertinent element in the assimilation of teachers (and students), and one that may benefit from a greater degree of acknowledgement and attention today than it was granted in Israel 20 years ago. Secondly, this case is an example of the potential rewards of a cultural influence that is mutual rather than one-sided. The contribution of Israel's immigrant teachers to the evolution of its math education culture suggests that the arrival of an immigrant population, though its values and priorities may be drastically different, can be seized as an opportunity for adapting and enhancing a pre-existing culture.

## **References**

- Amit, M. (2000) . Teaching Mathematics in a Diverse Society. In A. Ahmed & H. Williams & J. M. Kraemer (Eds.), *Cultural Diversity in Mathematics Education*, pp. 385-391. Chichester Horwood Publishing Limited.
- Amit, M. & Burde, I. (2005). Kenote presatation in the CIEAEM 58 conference, Italy.
- Amit, M. & Fried N.M. (2002). High Stake Assessment as a Tool for Promoting Mathematical Literacy and Democratization of Mathematics Education. *Journal of Mathematical Behavior*, 21. pp 499-514.
- Amit, M., Fried, N.M. & Abu-Naja, M. (2007). The Mathematics Club for Excellent Students as Common Ground for Bedouin and Other Israeli Youth. In B. Sriraman (Ed). *International Perspectives on Social Justice in Mathematics Education*, The Montana Mathematics Enthusiast, Monograph1 (pp 75-90). The University of Montana Press
- Amit, M. & Fried N.M. (2008). Research, Reform and Times of Change. In L. English (Ed.), *Handbook of International Research in Mathematics Education*, pp. 355-382. Mahwah, New Jersey: Lawrence Erlbaum Associates Inc. Publishers.
- Bishop, J.A. (1988). *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. London: Kluwer Academic Publishers.
- Darr, A. & Rotschild-Shaked, L. (2004). Social capital and the absorption of immigrant scientists and engineers into Israeli Communities of Experts. *Israel Studies* , 9.2, pp. 106-120.
- Dunstan, J. & Suddaby, A. (1992). The Progressive Tradition in Soviet Schooling to 1988. In J. Dunstan (Ed.), *Soviet Education under Prestroika*,. London and New York: Routledge. pp. 1-13
- Muckle, J. Y. (1988). *A Guide to the Soviet Curriculum; What the Russian Child is Taught in School*. London & NY: Croom Helm Publishing.
- Seah, W.T. (2004). *The Negotiation of Perceived Value Differences by Immigrant Teachers of Mathematics in Australia*. Unpublished Ph.D. dissertation, Monash University, Victoria, Australia. Available at the website: [www.education.monash.edu.au/contacts/staffDetails.php?uid=wseah](http://www.education.monash.edu.au/contacts/staffDetails.php?uid=wseah)
- Sfard, A. & Prusak, A. (2005). Telling Identities: The Missing Link Between Culture and Learning Mathematics. In H. L. Chick & J. L. Vincent (Eds), *Proceedings of PME 29*, Melbourne, Australia.

<sup>i</sup> Miriam Amit, Ph.D  
Prof. of Mathematics Education  
Head, Dept. and Center for  
Science and Technology Education  
Ben-Gurion University of the Negev  
Beer Sheva, Israel, 84105  
Tel:972-8-6461901 Fax:972-8-6472847

<sup>ii</sup> This paper is based in part on a presentation by Amit and Burde at CIAEM in Italy, July 2005.

<sup>iii</sup> It is worth noting here for the purposes of clarification that as Jewish immigrants entering Israel, these teachers benefited from the security of an automatic and immediate citizenship, and of the highly organized system the state employs for the integration of immigrants (there is an entire government office devoted to this). The year-long course was therefore necessary specifically and exclusively for obtaining a teaching certificate, and was not a prerequisite for any other aspect of Israeli citizenship.

<sup>iv</sup> Though we address only elements pertaining specifically to mathematics education here, these are largely derivative of the very great differences that exist between these cultures on a general systemic level as well.

## **Climbing and Angles: A Study of how two Teachers Internalise and Implement the Intentions of a Teaching Experiment**

Anne Birgitte Fyhn<sup>1</sup>  
University of Tromsø, Norway

*Abstract: In this innovative teaching experiment, the context of climbing is used to induce the teaching and learning of angle concepts. This article reports on the outcomes of a three day teaching and climbing experiment and the impact of this experience on the teacher's understanding of meso and micro embodiments of mathematics, angle representations, as well as shifts in their attitudes about teaching/learning geometry.*

**Keywords:** angles; embodied mathematics; geometry; guided un-earthing; mathematisation; mathematical archeology; mathematics and physical education; meso space; meso space representations; reflective research practice; teacher beliefs; teaching and learning

### **Introduction**

Based on previous work in which a twelve-year-old girl discovered angles in her climbing experience, Fyhn (2006) posited that climbing discourse can be a possible resource in the teaching of angles in primary school. Consequently the girl's class was introduced to the physical activity of climbing, as an integrated part of the teaching of angles (Fyhn, 2008). The class and two teachers took part in a three-day teaching and climbing experiment (TCE). The first day was spent at a local climbing arena with a focus on angles and climbing, the second day was half day of follow-up work at school (ibid.), and the third day was a follow up climbing-and-angles day three months later.

Innovative research-based teaching is of little use unless teachers internalize and implement it. There is an entire body of research in teacher beliefs that supports the previous statement. The focus of this paper is a presentation of the TCE's intentions and how these intentions were internalized and implemented by the two participating teachers. The analyses aimed to search for regularities in each of the two different teachers' development. The main research question was: 'How do two different teachers internalise and implement the students' mathematising of climbing as an approach to the teaching of angles?' The term *mathematising* is used as by Freudenthal (1973), 'mathematising something' means learning to organise this 'something' into a structure that is accessible to mathematical refinements. The two teachers' development is compared to the researcher's own development towards this approach to teaching. Schoenfeld (1998) claims that teachers' knowledge, beliefs, and goals are critically important determinants of what teachers do and why they do it. So these three aspects are discussed as well: 'How are the teachers' beliefs related to their goals and knowledge?'

---

<sup>1</sup> Førsteamanuensis Matematikdidaktikk  
Lærerut. og pedagogikk  
Universitetet i Tromsø  
anne.fyhn@uit.no

Regularities in the most different cases indicate robustness and possibilities for generalisation (Andersen, 2003). A comparative case study approach was chosen to explore this question. The TCE was designed and performed by the researcher.

### Theoretical Framework

According to Lakoff and Núñez (2000, p. 365) “Human mathematics is embodied; it is grounded in bodily experience in the world.” They further claimed that angles existed in the early geometry paradigm where space was just the naturally continuous space in which we lived our embodied lives. The naturally continuous space is unconscious and automatic (ibid.). This supports work on the angle concepts in primary school as an integrated part of the students’ physical activity. Berthelot and Salin (1998) claim that space could be conceptualised into three different categories:

microspace (corresponding to the usual prehension relations), mesospace (corresponding to the usual domestic spatial interactions), and macrospace (corresponding to unknown city, maritime or rural spaces...) In consequence, the space representation produced by the usual out-of-school experiences is not naturally homogenous, and is quite different from elementary geometry. (p. 72)

One goal of the TCE was to guide the students to build bridges between their embodied meso space climbing experiences, and the part of school mathematics that concerns angles. The research focus was also whether and how the participating teachers attained this goal.

#### *Inductive and Deductive Teaching*

In Norway mathematics traditionally is taught deductively. Alseth, Breiteig and Brekke (2003) claimed that Norwegian mathematics lessons usually start with the teacher’s explanation on how to solve a particular task. Then the students work individually on solving similar tasks in their books.

The curriculum of 1987 (KUD, 1987) was interpreted to recommend deductive as well as inductive mathematics teaching even for the lowest grades: “The subject matter may be introduced at first by the pupils’ investigating and experimenting in well prepared learning environments, and/or by the teacher showing and explaining.” (p. 195, author’s translation)

In the following curriculum of 1997 (KUF, 1996a; KUF, 1996b) the paragraph ‘Approaches to the study of Mathematics’ focused on learning; the students’ experiences and practical work. In this curriculum “practical situations and pupils’ own experiences” played an important role throughout elementary school. Despite the claim that students construct their own concepts, this curriculum too could be interpreted to support a deductive approach to mathematics teaching. The 1997 curriculum was also very vague regarding inductive versus deductive approaches so that it can be interpreted that the curriculum makers were not aware of these two different kinds of approaches. The 2006 curriculum’s (KD, 2006) focus was that during each of the main stages the students should aim to achieve some specific competencies in the main mathematics areas. This curriculum’s intentions focused on students’ achievements and did not concern teaching. However, the TCE took place before this curriculum was implemented.

In the U.S.A, teachers meet different requirements than the Norwegian ones. The National Council of Teachers of Mathematics, *NCTM*, standard includes explicit demands regarding inductive and deductive geometry teaching; “in grades 6-8 all students should create and critique inductive and deductive arguments concerning geometric ideas and relationships” (NCTM, 2000). However the author is cognizant that the standards espoused by the NCTM do not constitute a national curriculum and are viewed merely as recommendations.

In 1973 Freudenthal (1973, p. 402) warned, “The deductive structure of traditional geometry has not just been a didactical success.” Many Norwegian mathematics teachers have no theoretical bases for designing inductive teaching and in addition they lack experience of teaching and learning geometry inductively. There are no requirements for inductive approaches to mathematics in the Norwegian Curriculum, opposed to for instance the NCTM standard. The TCE had a clear inductive approach to teaching, and therefore the Norwegian deductive teaching tradition had to be taken into account in the analyses of how the teachers internalized and implemented the TCE’s intentions.

#### *Concretising and Mathematising*

Freudenthal (1983) described Bruner’s triad *enactive, iconic, and symbolic*:

enactively the clover leaf knot is a thing that is knotted, iconically it is a picture to be looked at, and symbolically it is something represented by the word “knot” (p. 30)

He further claimed that this schema might be useful in work with concept attainment (ibid.).

Concretising is often used deductively in Norwegian mathematics lessons as a tool for explaining something to students who do not understand what is being taught. However, Freudenthal (1983) claimed not to teach abstractions by concretising them. He advised to use the converse approach, i.e., to start off by searching for a phenomenon that might lead the learner to constitute the understanding of angles (ibid.). He further pointed out that “angles, in contradiction to lengths are being introduced and made explicit in an already heavily mathematised context” (p. 360). In the TCE the students mathematised their climbing by identifying and describing different angles related to climbing. Furthermore they were asked to explain their climbing moves via the use of the described angles. So the teaching of angles did not take place in a heavily mathematised context.

#### *Mathematical Archaeology*

The term mathematisation (Freudenthal, 1973) is to a large extent the same as mathematical archaeology. But while mathematisation refers to building of knowledge and not to discovering anything, the word archaeology refers to something hidden that needs to be uncovered. Mathematics can be integrated into an activity to such a degree that it disappears for both the children and the teachers, and then there might be the need for making the mathematics explicit. A mathematical archaeology is an educational activity where mathematics is recognised and named. This involves being aware that some activities carried out in the classroom are in fact mathematics.

An aim of a mathematical archaeology is to make explicit the actual use of mathematics hidden in the social structures and routines. It is the process of digging mathematics out and drawing attention to how mathematics moves from being an explicit guide to becoming a grey eminence underlying, for instance, social and economic management. (Skovsmose, 1994, p. 95)

It is important to a project, which contains mathematics as an implicit element, to spend some time on mathematical archaeology. The reason is:

If it is important to draw attention to the fact that mathematics is part of our daily life, then it also becomes important to provide children with a means for identifying and expressing this phenomenon. (p. 95)

It makes a difference whether the teaching is built upon situations that contain possibilities for the application of mathematics or just for descriptive purposes. The mathematics in climbing is so implicit that it is invisible (Fyhn, 2006). One goal of the TCE was to provide students a means for identifying, describing and using angles as an integrated element in their climbing activity, and consequently mathematical archaeology was an important part of the project. The mathematics here was descriptive.





Figure 1. Teacher Frode: “They (the students) managed to ascend the climbing wall, and from different positions they named angles in their bodies. For instance our elbow can shape a right angle.” The idea is first to let the students identify angles in a climbing context. Second, they describe these angles, and third, they explain how the described angles influence their climbing. The teaching aims to guide the students to use ‘angle’, as a tool for improved climbing technique.

Torkildsen (2006) denoted performing mathematical archaeology on a subject as the *un-earthing* of mathematics in this subject. In the TCE, the researcher intended to guide the students through a kind of guided re-invention (Freudenthal, 1991), where the focus was on the un-earthing of descriptive mathematics. This teaching was denoted as *guided un-earthing*. The students’ mathematising of climbing with respect to angles, will be explained as the un-earthing of angles in climbing.

#### *Different Approaches to Angles*

Freudenthal (1983) recommended to “introduce angle concepts in the plural because there are indeed several ones; various phenomenological approaches lead to various concepts though they may be closely connected” (p. 323). He (ibid.) distinguished between angle as a static pair of sides, as an enclosed planar or spatial part and as the process of change of direction.

Mitchelmore and White (2000) found that the simplest angle concept was likely to be limited to situations where both the sides of the angle were visible; it is more difficult for children to identify angles in slopes, turns and other contexts where one or both sides of the angle are not visible.

Henderson and Taimina (2005) pointed out three different perspectives from which we can define angles: as a dynamic notion, as measure and as a geometric shape. Angle as shape referred to what the angle looks like; namely angle as a visual gestalt.

Krainer (1993) divided angles into four different categories: angle without an arc (angle as linked line/knee), angle with an arc, angle with an arrow (or oriented angle space) and angle with a rotation arrow (angle describing the rotation of a ray).

The TCE intended to let the teachers experience the guided un-earthing of angles in a climbing context. Figure 1 shows one of the climbing students with bent joints both in her knees, heels, hips, shoulders and elbows. The TCE referred to three different levels of understanding angles (Fyhn, 2008). At the first level students recognise angles as bent bodily shapes. These are mesospace angles with neither arcs nor arrows, and the students are not asked about these angles’ sizes in degrees.

At the second level the angles are described, either by what they look like (acute - right - obtuse), or by a drawing or by a rope demonstration. The right knee of the girl in Figure 1 shapes a right angle while her left heel shapes an obtuse angle. Angles can be

described by both meso- and microspace representations. At the third level angles are a tool for improved climbing technique; it is harder to ascend a climbing route if you cling to a handhold with your elbow in a right angle position, than if the angle is obtuse (ibid.).

#### *Approaches to Angles in the Norwegian School*

Johnsen (1996) warned that the most frequently used way of working with angles in Norwegian schools was measurement, and she further claimed that a large amount of Norwegian primary school students used the protractor incorrectly.

The Norwegian curriculum of 1997 (KUF, 1996a; 1996b) focused on students' experiences and their conceptual understanding. But regarding angles, 4<sup>th</sup> grade students were to gain experiences with 'important angle measurements' (KUF, 1996a, p. 162, author's translation). However, the English translation of the curriculum (KUF, 1996b) said 'important angles'. The curriculum text further continues "especially a whole turn as 360 degrees, a half as 180 degrees and a quarter as 90 degrees". This indicated a continuation of the measurement approach to angles in Norwegian primary school.

The curriculum of 2006 (KD, 2006) pointed out a clear measurement approach to angles: The word *angle* occurs only once and that is under the subject area 'measurement' for students at the fourth grade: "An aim for the teaching is that the student... is able to estimate and measure... angles" (ibid., p. 28, author's translation). According to Van den Heuvel-Panhuizen (2005) "Measurement and geometry are two domains, each with their own nature." (p. 13). In the curriculum of 2006 (KD, 2006) measurement and geometry occurred as two different sections, but angles are only treated in the section measurement.

In the Norwegian KIM project in geometry (Gjone and Nortvedt, 2001) more than one third of the participating sixth grade students were consistent in their reasoning about why a small angle with long sides is larger than a larger angle with shorter sides. This indicated a need for a new approach to the teaching of angles in Norwegian schools; neither Johnsen (1996) nor Gjone and Nortvedt (2001) could be interpreted to support the established measurement approach.

#### *Teachers' beliefs*

Törner, Rolka, Rösken and Sriraman (2010) paid attention to Schoenfeld's *Teaching-in-Context* theory, which pointed out interdependencies between the three fundamental variables knowledge, goals and beliefs, the *KGB* variables. "A teacher's spontaneous decision-making is characterized in terms of available knowledge, high priority goals and beliefs" (ibid., p. 403). Teaching here is understood as a continuous decision making process, and these three variables are considered as sufficient for understanding and explaining numerous teaching situations (ibid.). Lerman (2002) points out the cyclical relationship between changing beliefs and changing practices. Because one of the informants in the TCE is a trainee, it is of less value to research the two informants' change in practice. But the relations between their goals, beliefs and knowledge are visible to a large extent. So the analyses in this paper will relate Therese's and Frode's beliefs to their goals and previous knowledge.

#### **Methodology**

The TCE was closely related to design research (Gravemeijer & Cobb, 2006). In design research the designed teaching experiment undergoes iterative cycles of refining, while the TCE represented only one cycle. The TCE research focus was the process of teacher development.

The two participating teachers, Therese and Frode, were quite different people with different backgrounds; Therese was a trainee while Frode was an experienced mathematics teacher. When they joined the TCE both of them were acquainted with climbing and with mathematics teaching, even though their competencies and experiences differed to a large extent. Both Therese and Frode had had experienced inductive teaching, but these experiences mainly concerned science and other subjects. Frode had a couple of experiences with inductive mathematics teaching while Therese had none.

In addition to the teachers and the researcher, one more trainee and two other skilled grown-ups also took part in belaying<sup>1</sup> climbing students at day one, to make sure that as many students as possible could climb at the same time.

### *The Class*

The entire class consisted of 18 students in the seventh grade. For different reasons some students were absent from different parts of the TCE. Nine girls and four boys in seventh grade participated the first day, the entire class participated the second day, while nine girls and six boys participated the third day.

One week before the TCE the students performed a pre-test with tasks that focused on angles and geometry. In this test more than half of the participating students failed in a KIM-test task where they should pick out the largest and the smallest among five given angles (Gjone & Nortvedt, 2001). This indicated that the students' conceptions of angles needed improvement (Fyhn, 2008).

### *The Researcher*

The researcher designed the TCE and directed the completion of it, while the two teachers were assistant participants. The researcher's ability to let students mathematise their own climbing activities through performing mathematical archaeology was the result of a five-year long unguided process while she was teaching mathematics for trainee teachers.

The researcher's starting phase included three different aspects; firstly, an increased use of inductive teaching by use of artificial concretisations. Secondly, she performed meso space activities as basis for the teaching (Fyhn, 2002a), and finally, she performed some mathematical archaeology herself (Fyhn, 2001a; 2001b). The second aspect, meso space activities, turned out to make use of inductive approaches.

	Abstract symbols. Deductive approach	Artificial concretisation. Inductive approach
Micro space	A	B
Meso space	C	D

Table 1. Four different categories of geometry teaching. Category A shows the traditional Norwegian teaching, while the researcher's starting phase is presented in category D.

Table 1 presents four categories of approaches to teaching; the traditional deductive approach A and the researcher's inductive meso space approach D, where artificial concretisations were used. Table 2 presents four categories of mathematics, where E represents the mathematics that has been found in Norwegian curriculums until 2007, and H represents the mathematics that needs un-earthing in order to be described explicitly. The TCE focus on guided un-earthing of mathematics, was based upon the category H in table 2.

	visible	invisible
Construction by ruler and compass. Calculations. Proofs. Measurement. Manipulation of symbols.	E	F

Descriptive mathematics	G	H
-------------------------	---	---

Table 2. Four different categories of mathematics. Categories G and H represents the descriptive use of mathematics which is new in the 2007 curriculum (KD, 2006), which was implemented after this study took place. Invisible mathematics has never been explicitly focused in the Norwegian curriculum.

On one occasion the researcher succeeded in performing guided un-earthing of mathematics together with a fellow teacher educator (Mathisen & Fyhn, 2001). However, when a follow-up study was planned with two more colleagues, the result reflected several examples of what Skovsmose (1994) denotes as *artificial concretisations* of mathematics; category D in Table 1. In addition there was one single example of mathematical archaeology performed by the researcher; a focus on category H in table 2, and no more examples of guided un-earthing (Fyhn, 2002b). This is explained by the prematurity of the researcher's own ideas at that point in time. In 2004 the researcher finally succeeded in performing guided un-earthing of mathematics (Fyhn, 2006), and this time she was able to give a better description of what she did. On this occasion the participants were just the researcher and one single informant, and therefore the risk for interference from other people was minimised.

In planning the TCE, the researcher needed to avoid two possible events: a) the TCE could end up as a 'meso space artificial concretisation' project (category D in table 1) and b) a focus on category H where the teachers, and not the students, were performing the mathematical archaeology. Based on this, the researcher decided the teachers' roles in the project to be participant observers.

#### *The teachers*

By taking part in the project lead by the researcher, the teachers were provided with the experience of a *guided un-earthing* of angles in climbing. Even though both the teachers were familiar with mathematics and climbing, their competencies and experiences differed to a large extent. Both of them had studied one full year of university mathematics. Therese was a female professional climbing instructor who was a novice mathematics teacher, while Frode was a male experienced mathematics teacher who was a novice at climbing. They both had the qualifications for teaching mathematics at high school level in Norway. Therese's father was a scientist who lived in the Norwegian capital, while Frode's parents ran a small goat farm in the countryside in the northern part of the country.

Frode enjoyed taking part in physical leisure time activities and he was also partly responsible for his class' lessons in physical education. As for climbing, Frode was familiar with belaying climbers, and sometimes he went climbing himself. So he took active part in belaying the students.

Therese was an International Mountain Guide, and because of her climbing skills she was responsible for the security while the students were climbing. Being the researcher's trainee, she got some limited teaching tasks for each day of the project. Therese discussed these tasks with the researcher at the beginning and at the end of the TCE days.

The students performed the pre-test and a similar post-test in Frode's lessons, and he marked the students' answer to these tasks. The researcher set up these tests and handled them to Frode.

#### *The three Days*

On day one, before the climbing started, the day's two focus-words, 'climbing' and 'angles', were written in bold letters on a flip-over and the students were reminded of this throughout the day. The climbing was top-roping; the rope goes from the climber's harness up to a carabiner in the ceiling and down again to another person. This person is belaying the

climber; the belayer keeps the rope to the climber tight at all time, to prevent the climber from reaching the floor if she or he falls.

After having been introduced to some examples, the students had to mathematise their own climbing by identifying and describing different angles shaped by their bodies, the ropes and the building. Assisted by the teachers, the researcher guided the students through this *mathematical archaeology* on climbing with respect to angles.

In the day's last lesson a meso space perpendicular bisection was constructed on the floor by use of a climbing rope, a chalk and the bodies of some students. The location was the climbing arena. The activity made use of a rope, which is an element in the local climbing context.

The aim of this day was to let the students transform their ideas from working with angles in meso space into working with angles in micro space; an approach towards abstraction both by use of words as symbols and through construction by ruler and compasses. The students shaped angles by using their own bodies, they studied and discussed how the rope passed through the belay device, and they made drawings from the climbing. The students were split into groups, and each group constructed the perpendicular bisection on the floor by use of rope and chalk before they constructed it with ruler and compasses in their books.

In the period between day two and day three, Frode tried to implement some mathematics into his physical education lessons. Therese and the other trainee presented their impressions from taking part in a researcher's field work, to the rest of their fellow trainees.

The aim of the third day was to provide the teachers and the students with time for reflection, and then to re-visit them after some months to see if any change had occurred. The students were divided into groups, and each group had to create one particular climbing route; they decided rules for which holds they were allowed to use. The groups got small pieces of coloured cloth to mark their holds; each group got their own colour. Afterwards the groups would describe how their routes were ascended, and these descriptions had to include something about angles.

### *The Data*

Each of the three mornings the teachers wrote down their expectations, and in the end of each day they wrote down their impressions. In December, after the two first days, the teachers were interviewed on tape. In May, they got an e-mail asking their opinions about the use of climbing as basis for teaching about angles in primary school. In the end of June, each of them was visited by the researcher, in order to go through what was written about them so far. The intention was to make sure that their writings were interpreted as correctly as possible; to make sure that the English version of the collected and analysed data reflected their real opinions. However, maybe the teachers' experiences from the TCE made them change their minds, if so there could be some difficulties in validating the data.

In addition to the formal writings, some e-mail and sms correspondence took part when the researcher felt a need for contact, but this informal communication was not treated as data. The use of video in this study could have offered better possibilities to return to what exactly happened. Then there would have been more possibilities of analyses of the data and of restudying details too. But then the researcher's written material would have been an interpretation of what the teachers expressed in these videos. The focus of this research was whether and how the teachers internalized and implemented the intentions of the TCE, and video is considered not to be the best tool for getting valid data about this. Most of the data in this research was the teachers' own written material and that made the analyses close to the data. One aim of this paper was to focus on the teaching of angles and not on the teachers' beliefs. But the teachers' interviews and their e-mails showed that their beliefs and attitudes

were interwoven in their replies. Thus some attention is paid towards this aspect as well. The data in this study was:

- The teachers' writings about their expectations to and experiences from each of the three days
- The interviews with each of the teachers after day two
- Frode's e-mail about how he worked out an idea that he talked about in the interview
- Field notes from the presentation Therese and the other trainee held for their fellow trainees
- The teachers e-mail replies to the question "*Can climbing be used as basis for teaching about angles?*"
- The teachers' comments to the researcher's analyses of their writings
- The researcher's list of publications from the years 2001-2006

## Analyses

### *The First Day*

At the beginning of the day, Therese believed there was a great potential for angle teaching based on climbing. Frode expected both the students and the teachers to learn a lot about angles. Therese's expectations were categorised as uncovering invisible mathematics; categories F or H in table 2. Frode's expectations were more difficult to categorise.

In their writings at the end of the day, neither of them mentioned mathematising of climbing nor angles. But both of them filled about half of their lines with appreciation of the perpendicular bisection construction, which is interpreted as 'meso space artificial concretisation', category D in table 1. Therese lost the angle focus when she started focusing on belaying, while Frode did not mention the word angle in his text.

The perpendicular bisection activity far from fulfilled their expectations from the beginning of the day. This can be interpreted as that 'meso space artificial concretisations' corresponds with a view of mathematics that is found in category E in table 2. Maybe the teachers just claimed that they appreciated to experience some inductive meso space mathematics teaching; that could be interpreted to that they had reached what the researcher describes as her starting phase.

### *The Second Day*

The aim of this day was to provide the students with follow-up work at school, and bridge the gap between their meso space experiences from the climbing wall and their micro space work with pen and paper. The students might mathematise their climbing with respect to angles through practical activities, climbing talk, drawings and oral discussions. Because the artificial concretisation of the perpendicular bisection showed to be a very popular activity, and belaying showed to be a very popular activity among the students, some extra attention was paid towards these two activities. The students' mathematising of the belay device's functioning with respect to angles would indeed fulfil the TCE intentions because it was descriptive use of apparently invisible mathematics; category H in table 2.

Therese put on a harness, attached herself to a rope by a belay device, and asked what to do if she was belaying someone who fell. She asked for angles shaped by the rope and the belay plate. But the students did not understand what she meant. Therese concluded that she should rather have let the students perform this activity themselves, and then more of them probably would have understood what she meant. She indicates the difference between whether the teacher or the students perform the *un-earthing* of angles; if the students had

performed the activity themselves, they could have identified and explained angles by trial and error, by repeatedly checking out how different ways to use the device worked out.

Frode was busy doing various other things so he did not write about his expectations and experiences this day. Unfortunately the researcher was not aware of this until afterwards. This is an example of the data's weaknesses, and these weaknesses need to be sorted out. Therefore there could not be pointed out any similarities between Frode's and Therese's writings from this day.

Most of Therese's expectations concerned mathematics. She was curious about how much of the students' understanding of angles there would be left from day one. She was curious about how the groups would succeed in the construction of the perpendicular bisection. She ended: "I believe I will learn about how to work with concepts in the classroom with angles as starting point." This can be an indication of some expectations about some further mathematics, beyond the results of the descriptive mathematical archaeology.

At the end of the day she wrote that she was satisfied and pleased about how much the students had absorbed about angles:

A physical approach to angles leads some misconceptions to surface. The students are not sure which angle we refer to. Many of them thought that the angle disappeared when the rope was straightened. And that is correct in a way. But I believe they absorbed that the straight rope represents a  $180^\circ$  angle. The students really differ in how fast they understand this. But with this approach I believe that we reached all the students at some level, and that all of them have got something from this.

Therese here nicely described how the students' conceptions of angles were extended because their intuitive ideas of angles were challenged while they tried to understand how to belay a fellow climber. Here Therese experienced a *guided un-earthing* of angles; the students' un-earthing of apparently invisible mathematics for a descriptive purpose. These students' mathematising of climbing with respect to angles caused extension to their conceptions of angles.

### *The Interviews*

The 'meso space artificial concretisation' (category D) experience of the perpendicular bisection construction fulfilled the expectations that the teachers presented in the interviews. Frode had experienced practical mathematics teaching that took place outside the classroom; categories C or D, while Therese had experienced mathematics teaching that differed from the traditional deductive teaching she was used to; categories B or D.

Therese appreciated observing the students' growing consciousness about angles in their bodies:

The students said that, well... there are no angles in our bodies... and then the consciousness-raising that happens throughout such a day... On the second day, when they were asked to perform an acute angle and a right angle by their bodies, then we could see all these different ways to stand and move. That was nice.

Her description here was interpreted as students' un-earthing of angles; a move from category H to category G. When she was asked if she thought that the students would think about angles related to climbing in the future, she answered that they would have to put their 'angle glasses' on. This statement was interpreted that she thinks the students can use their climbing bodies as models for angles. Furthermore she said:

Then the natural activity can take its own course, but the mathematics is still there. I like that. If the subject is all about mathematics I believe there will be some impatience, because you do not get the natural flow that we had on that particular day. I really appreciate the balance we got that day, to get the mathematics in while they performed activities ...and talked about it ...and related it and associated it to mathematics, yes ... that is more natural.

According to Therese the students' basic knowledge on the second day differed from their basic knowledge on the first day; angles seemed to concern them in a way.

Further on in the interview she pointed out that the first day's focus was on climbing while the second day's focus was on mathematics, and she appreciated "the natural progression to get more focus on the subject." Therese's utterances could be interpreted as a reference to the *guided un-earthing* of angles in climbing. Frode's utterances did not indicate a similar focus.

According to Frode the students' attitude towards mathematics was very negative when he started teaching them this autumn. He was not sure if he has managed to change any of this, but many of the students had started claiming that they enjoyed mathematics.

When Frode was asked if he believed that the students would be relating angles to climbing in the future, he answered: "We have worked with angles related to climbing. I believe that in future talk about angles the word climbing will show up, and consequently they will think about what we did." This way of connecting angles to climbing is interpreted as the students' use of climbing bodies as model for angles. Therese made a similar claim about models, and they both are interpreted to believe that the students' will remember the move from category H to category G in Table 2.

### *The Teachers' Attitudes*

According to Lerman (2002) "It is in the recognition of conflict between what one wishes to do, or believes oneself to be doing, and the perceived reality of one's teaching that can bring about change"(p. 234). Törner, Rolka, Rösken and Sriraman (2010) support this by pointing at the relationship between goals and beliefs. The interviews showed that the teachers recognised such conflicts, and consequently they had a positive attitude towards participating in the TCE. The focus in this paper was on the teaching of angles and not on the teachers' beliefs, but it turned out that some beliefs came to surface.

Frode's school focused on 'outdoor schooling'; schooling outside the ordinary school building. His school even offered guiding in how to use outdoor schooling in theory and in practice. Frode wanted to be loyal to his school's aims. However, if he asked his colleagues for the mathematics content in their outdoor schooling, they answered that you could take the children to the shore and count stones and pebbles.

I think you can do that with students at the lower grades... You have to look ahead, try something... that is where I feel I really need something more. How to use outdoor schooling in mathematics teaching for students at 7th and 8th grade?

Frode had a conflict between his personal goals and what he experienced in his classroom, and consequently he was ready for a change in teaching practice (ibid.). He probably had two reasons for wanting to improve his teaching. Firstly, with respect to the practical work that ran like a connecting thread throughout the curriculum. Secondly, with respect to 'outdoor schooling' in the way it was focused on at his school.

Therese's statements described a conflict between her personal goals and her experiences from the mathematics classrooms:

I became a mathematics trainee teacher just to get the paper that proves I am a teacher. Mathematics is a subject that I had left far behind; actually I am really fed up with it. Now that I have started teaching myself, I find myself in the worst case scenario regarding mathematics. I experience my own teaching as dreadful and boring... This goes deep into my soul... I really do not want to force this upon other people because I think this is not all right.

Therese was interpreted to have two reasons for her positive attitude towards the TCE. Firstly, she wanted to experience mathematics teaching that was based on students' mastery experiences. Secondly, she wanted to experience mathematics teaching that differed from the deductive approach of which she disapproved.



Both of the teachers' attitudes were interpreted as that they wanted something that differed from deductive micro space teaching (category A in Table 1).

#### *The Trainees' Presentation*

A few weeks later, Therese and the other trainee did a presentation to their fellow trainees. They were free to choose among everything that happened on day one. They could have chosen to focus on the climbing approach to angles, by for instance letting their fellow trainees mathematise some belay devices. However, they chose to demonstrate the meso space perpendicular bisection.

There could be several reasons for their choice; maybe the other trainee's view of mathematics teaching to a great extent corresponded with a 'meso space artificial concretisation' (category D) approach to some visible (category E) mathematics, and therefore she did not see the point in the *guided un-earthing* of invisible (category H) mathematics. Maybe Therese was misled by listening to her well-meaning fellow trainee, just as the researcher was by listening to her well-meaning fellow teacher educators, on her way towards an idea of how to perform *guided un-earthing* of descriptive mathematics.

Another possibility was that the other trainee just believed in climbing as a social and exciting activity, and that the researcher's enthusiasm for mathematising of climbing had influenced and overwhelmed both of them. Maybe the other trainee was just as convinced about the climbing approach to angles, as the audience who was able to see "The Emperor's new clothes". Then the perpendicular bisection would represent an acceptable mathematics alibi.

Some months later Therese was asked why she let their fellow trainees do just the perpendicular bisection. Without hesitating she answered, that this was a simple and practical task that was easy to perform in the actual room, and that they had had some discussion before they decided what to do. She continued, "By activating the other students we could describe how to do mathematics in a practical way, and we illustrated how you wanted to work with mathematics." This claim can be interpreted to be that she thought that the researcher's aim was practical work just like what the curriculum points out. Therese was interpreted to be in a phase similar to the researcher's first phase.

#### *Frode's Teaching Practice*

During the interview, Frode suggested to integrate some mathematics into his physical education lessons. A couple of weeks later, he was e-mailed and asked how this had worked out. Frode had tried to keep mathematics in the back of his mind throughout two physical education lessons with the class. He had used the words 'line', 'velocity', 'angle' and 'direction' in his instructions, "many of the concepts are mathematical, but many were everyday concepts which we regularly use in everyday language." Regarding physical education as additional subject he wrote:

We do many coordinating exercises where especially the angle concepts are used; usually the students do somersaults in many different ways. I see that we can use the concept of rotation here. I give instructions to the students like that "*in your next jump you shall rotate horizontally 360°*"

Frode's descriptions of these lessons show that he is performing mathematical archaeology on his physical education lessons to a large extent, he has a focus on un-earthing the category H mathematics. Furthermore, he chose a deductive meso space (category C) approach in teaching his recently acquired descriptive view of mathematics (category G) as part of his physical education teaching. This could be interpreted to mean that Frode was in a similar phase as the researcher was, when she was performing mathematical archaeology on her students' work (Fyhn, 2001a; 2001b).

Frode wrote, “he he, you have opened my eyes a bit here”. This was a strong statement, which was interpreted as that he had internalized and implemented some of the intentions of the TCE. “It is all about possibilities, not limitations”, he wrote. Frode was becoming acquainted with *un-earthing* mathematics from his students’ physical experiences, but there was no sign of guided un-earthing.

### *The Third Day*

In the morning, Therese and the students started out by writing about their memories from the first two days, and their expectations to this day. Unfortunately Frode was prevented from taking part in this writing session. This lack of data made Therese’s writings less useful because they could not be compared to Frode’s writing. The last time Therese thoroughly learned that demonstrations are useful simply to create an image of something to copy; she claimed that to learn something the students need to have something in their own hands, and try it themselves afterwards. In addition she had learned new ways of thinking about angles. Her text is interpreted to be that she disapproves of deductive micro space teaching (category A).

Therese had no expectations concerning mathematics this day. She was curious about the day, and expected a nice day with possibilities for her to give some climbing advices to the students. Afterwards Therese enjoyed watching the students making routes and discussing what holds that were natural to use related to their movements. And “It seems as if the students’ conceptions of angle are more profound now than the last time.” Her writing can be interpreted as that the students’ improved conceptions of angles was caused by building of bridges between visible and invisible angles (categories H and G).

Furthermore Therese pointed out a misconception caused by language; the Norwegian word *rett* means both straight and right. “A straight leg with a 180° angle can quite easily be called a right angle in Norwegian. And that is not unnatural because that is what we connect with the word right.” This writing could be interpreted as a description of how she experienced that the *guided un-earthing* of angles in climbing helped the students to get over an expected misconception. Frode, however, did not make any claim about how to prevent possible misconceptions.

Most of Frode’s writing concerned mathematics and mathematising. He even claimed that the students were aware of un-earthed angles:

At the end of the day, during the presentations, I observed that the students had learned to put angles down in words. They managed to ascend the climbing wall, and from different positions they named angles in their bodies. For instance our elbow can shape a right angle.

His writing could be interpreted as students’ improved conceptions of angles was related to mathematical archaeology on climbing; “Mathematics has to be recognised and named” (Skovsmose, 1994). Freudenthal (1991, p. 64) claimed similarly, “Name-giving is a first step towards consciousness.” However, Frode’s words “...the students had learned...” did not indicate whether he believed that they learned it through guided un-earthing or because of meso space teaching (category C).

Furthermore Frode appreciated that the students had found out how to use a table as a tool for organising and structuring their information; the students were able to analyse their information after having put it into a table. This activity was the students own mathematising of climbing, but not with respect to angles this time; the students turned mathematics from invisible (category F) into visible (category E) by un-earthing it. This was the students’ own idea, and no guiding took place. What Frode described here was how the focus on mathematical archaeology seemed to generate original mathematical reasoning from the students. Therese did not mention this event.

### *Beliefs, goals and knowledge*

The *KGB* variables (Törner, Rolka, Rösken & Sriraman, 2010) will enlighten relations between the teachers' beliefs, and their knowledge and goals. Therese, who is a trainee, had no previous knowledge about mathematics teaching except for her experiences from being a student in mathematics classrooms. But she had expressed two clear goals with her participation in the TCE: i) to experience mathematics teaching that was based on the students' mastery experiences, and ii) to experience mathematics teaching that differed from the deductive boring approach she was used to. She had reached the goals, but her limited knowledge about mathematics teaching indicates that it probably is unrealistic that she would be capable to have implemented the TCE's intentions. But on the other hand, her claims were interpreted that she believed the TCE was a nice approach to teaching. That is not necessarily the same as that she was able to carry out a similar teaching herself.

Some of Frode's knowledge regarding mathematics teaching came to surface during the TCE. He was an experienced mathematics teacher with a solid background as for the subject mathematics. He was aware that teachers, who were less competent for teaching the higher grades, influenced the mathematics teaching at his school. His goal was to experience practical mathematics teaching outside the classroom. And this teaching had to correspond to the higher grades' syllabus. According to his writings, he had reached this goal. His students had been aware of un-earthed angles and that they had found out on their own how to use a table as tool for organizing their information. Both Therese and Frode may be interpreted that they believe that climbing might function as a basis for the teaching of angles.

### *A Couple of Months later*

The data could be interpreted to indicate that the teachers to some extent had internalized and implemented the intentions of the TCE. However, maybe the teachers did not want to disappoint the researcher, and consequently wrote what they believed she expected from them. So, the data needed careful validating, and the teachers were e-mailed some months later and asked to reply in two to ten lines: "May climbing be used as basis for teaching about angles?"

Therese's reply arrived less than two hours later. She started out claiming that there are lots of angles both in the climbing bodies and in the climbing gear for belaying. She argued that she found the adjusting of angles in arms and legs to be an element in the climbing moves. Her writing was interpreted to mean that the students were able to mathematise their climbing; that they were able to *un-earth* angles in their climbing experiences. Consequently Therese was interpreted to have internalized the TCE's intentions:

The climber, who is conscious about this, can feel it in her own climbing, and make active use of it as an element in the climbing technique. Good climbing technique is based on the least possible use of force. This is active thinking about angles.

Therese's text was interpreted to mean that mathematising of climbing with respect to angles is easy, because the climbing context is pervaded with static and dynamic angles both in the ropes, in the wall and in the climbers' bodily joints. However, she could not be interpreted to have implemented the intentions before she had tried to implement guided un-earthing in her own teaching. Maybe then she would end up like Frode, who seemed to be satisfied with performing the mathematical archaeology himself. However, she had got as far as possible for her at that given moment. In addition she pointed to the students' positive attitude toward this activity: "Most people experience climbing to be exciting and fun".

Therese did not clearly point out anything about the teacher's role regarding the *guided un-earthing* of angles in climbing. Her beliefs and goals here concern angles in climbing and not an inductive approach to teaching. As previously pointed out, Therese is a trainee, and consequently she knows almost nothing about inductive mathematics teaching.

But as a professional climbing instructor, she knows a lot about climbing, and her goals and beliefs regarding angles as an element in the mathematics teaching is coloured by this knowledge.

Frode's reply arrived three days later. He wrote 15 lines concerning his opinion about physical activity in school in general, "... Children enjoy physical activities, and so do adults. Physically active children are happy children!" followed by 11 lines where he focused on the question. His answer was yes, but 'experience' was the only reason he gave. There was a great risk in interpreting Frode's e-mail to be that he was not convinced about anything related to climbing and angles. Because the TCE was a comparative case study with only two informants, it was natural to make one more inquiry to investigate if this really was his answer.

But there was a risk that Therese's and Frode's e-mails did not reflect what they really meant about climbing as basis for teaching mathematics. Maybe their e-mails just revealed what they thought the researcher expected them to write or what they felt persuaded to write. They were asked to reply to a question which concerned the intentions of the TCE, but there was no guarantee of how they would interpret the question.

#### *A final Visit*

In the end of June, Therese verified most of the writings about her. Frode immediately pointed out that his last e-mail was meant as a start of some longer writing, but that this longer writing never was continued. So his last e-mail did not reflect what he actually meant.

Frode explained that when his students worked with time and velocity during this spring's mathematics lessons, they started with performing a 'running experiment'. They finished with making a written report that explained what they had done, and how they could find the average velocity. What he says here can be interpreted as the students' un-earthing of mathematics from their meso space activity. Frode did not claim whether his approach to teaching here was inductive or deductive, but he had guided the students to build a bridge between their embodied meso space experiences and school mathematics. He immediately made a new version of his reply to the question. At first he wrote that climbing was a great fundament for the teaching of angles,

Children use their bodies to shape different angles. This gives them a closer relationship to angles. The students in my class enjoy climbing, and after the climbing days some of the students said: '*Angles are fun!*' I believe the students will remember '*angles*' in their future climbing.

In addition Frode was interpreted to claim that angles would concern his students' future climbing activity. His belief here makes sense when related to his goal; to experience practical mathematics teaching outside the classroom. His knowledge and experience about mathematics and mathematics teaching was the background for his goal.

Frode also showed that his own teaching practice was changing. He and his students had un-earthed mathematics from their running this spring, and they had even made a written report about this. This is what Lerman (2002) claimed; a change in teaching practice is related to a change in belief. Frode was interpreted to have implemented the intention of performing mathematical archaeology on the students' physical activities. Both Frode and Therese were interpreted to claim that they have internalised and implemented some of the intentions of the TCE. However, the difference is that Therese sticks with the guided un-earthing, while Frode tends to perform the un-earthing himself and present the un-earthed mathematics to his students in a deductive way. This might be due to the teachers' knowledge and goals. Therese had no knowledge about inductive mathematics teaching but a lot of knowledge about climbing, while her goals focused on inductive teaching and mastery experiences. In the end she seemed to believe in guided un-earthing of the students mastery experiences from climbing. But she had not had any possibility of performing such teaching. Frode's goal

concerned mathematics teaching outside the classroom, and his change in practice is a strong indication to that his new belief concern un-earthing of mathematics from physical activities. Both of them show a development that has much in common with the researcher's development. The difference is that Frode is interpreted to be loyal towards a deductive teaching.

## Discussion

Wood and Berry (2003) underlined the importance of creating a shared knowledge base for teaching. They claimed that research on the process of development extends the idea of a 'product'; "the process involved can become the product that is sought" (p. 197). Regarding teacher development, they ask for reports of research that study the process-into-product models. The TCE intends to be such a report.

Leatham (2006) warned researchers against assuming that teachers easily can articulate their beliefs. He also pointed the simplistic thinking of there being a one-to-one correspondence between what teachers state and what researchers think those statements mean. Leatham's (2006) warning against the dangers of simplistic one-to-one correspondence between what teachers state and what teachers mean matches the TCE's analysis; as shown in Frode's first e-mail reply to the question about climbing as basis for teaching about angles. The main data source of the TCE was the teachers' written statements. The analyses of the teachers' written statements were presented to the teachers, in order to have the analyses as close as possible to what they really meant.

According to Brekke, Kobberstad, Lie and Turmo (1998) it had been problematic for Norwegian students to grasp that  $180^\circ$  is an angle. A strengthened rope represents a  $180^\circ$  angle where both of the sides are visible. At the end of day two, Therese wrote, "many of the students thought that the angle disappeared when the rope was straightened..... But I believe they absorbed that the straight rope represents a  $180^\circ$  angle." This writing indicated that the angles shaped by climbing ropes can represent a useful contribution to the teaching of angles; that students' mathematizing of the belaying of climbers could prove to be useful to extend the students conceptions of angles.

According to Gravemeijer and Cobb (2006) the Dutch *RME* (Realistic Mathematics Education) had emerged

in resistance to instructional and design approaches that treated mathematics as a ready-made product... A process of guided reinvention then...requires the instructional starting points to be experimentally real for the students, which means that one has to present the students problem situations in which they can reason and act in a personally meaningful manner.(p.15)

In the TCE, the students' conceptions of angle were treated as something the students created as an integrated part in the development of their climbing talk. None of the climbers asked why they had to climb, or what they needed these experiences for; this is interpreted to be that the students found the activity to be meaningful to them. According to van den Heuvel-Panhuizen (2003)

Models are attributed the role of bridging the gap between the informal understanding connected to the 'real' and imagined reality on the one side, and the understanding of formal systems on the other. (p. 13)

This corresponds to one intention of the TCE; to guide the students to build a bridge between their (embodied meso space) experiences and school mathematics. This matches Frode's claim, that he believes the students will remember angles in their future climbing.

## Findings and Conclusions

The main focus of this research was whether and how teachers internalised and implemented guided un-earthing of angles in climbing as an approach to the teaching of angles. The data concerning the teachers were compared to the development of the researcher's publications in order to search for common developmental features. Five years passed from the first time the researcher performed guided un-earthing (Mathisen & Fyhn, 2001) and until she managed to work it out the second time (Fyhn, 2006). The first time she hardly was able to describe the un-earthing, but the second time she had developed a tool for describing it explicitly. Through these five years, the researcher was easily misled into what Skovsmose (1994) denotes as *artificial concretisation* by listening to well-meaning fellow teacher educators. For a period she was even satisfied with performing the un-earthing of mathematics by herself, instead of guiding her students to perform it.

The findings indicate some regularity in the two teachers' development, and their processes of development are to a large extent similar to the researcher's development towards guided un-earthing. The TCE was a three day descriptive work with mathematics, with no explicit focus on problem solving or task solving. This was unclear to the researcher and therefore the informants were not informed about it. Many teachers do not treat descriptive work with mathematics as real mathematics (Skovsmose, 1994). The teachers' lack of knowledge about descriptive use of mathematics might have influenced their goals and beliefs about the TCE. According to Schoenfeld (1998) the questions what a teacher will do next, and why, can be illuminated by describing interactions between her knowledge, goals and beliefs.

There were strong indicators of relations between the teachers' knowledge, goals and beliefs. The trainee Therese's beliefs were restricted because of her limited knowledge about inductive mathematics teaching; her goals were to experience mathematics teaching that differed from the deductive micro space approach that she was familiar with. Frode's beliefs and goals were related to his knowledge about how his teacher colleges taught mathematics, and his beliefs were related to his knowledge about gymnastics teaching.

Before the TCE, none of the teachers were familiar with inductive approaches to teaching mathematics, but they were familiar with inductive approaches to teaching physics. At the end of day one, none of them mentioned the climbing approach to angles. However, both of them appreciated artificial inductive meso space teaching (category D in Table 1). They are interpreted to have entered a phase similar to the researcher's phase when she was trying to grasp mathematical archaeology and mathematising.

At the end of the second day, Therese nicely described how the students' intuitive ideas of angles were challenged, while they tried to understand how to belay a fellow climber. She is also interpreted to have experienced and appreciated a situation where *guided un-earthing* of angles caused extension to the students' conceptions of angles. She can still be interpreted to be in a phase where her implementation of the guided un-earthing of angles is premature or diffuse. Together with her fellow trainee, she chose to present their fellow trainees with some artificial inductive meso space teaching (category D in Table 1). They could as well have chosen to guide their fellow students in un-earthing of angles in the belay device's functioning. This interpretation indicated that Therese's development followed similar pattern as the researcher's development.

The interviews indicated that Therese and Frode both to some extent believed in the students' un-earthing of angles from climbing as an appropriate approach to the teaching of angles. Both the teachers were interpreted to claim similar utterances: The students had grasped that climbing bodies shaped angles, and that different bodily moves would shape different angles. But maybe their claims in the interview reflected just what they believed the researcher wanted them to say. According to Lerman (2002) it is a methodological weakness

to assume that interviews and questionnaires can reveal beliefs, which is the main determinant of a teacher's action in the classroom. This yields particularly Therese, who is a trainee and has no class on her own. At the end of day three the teachers' writings were interpreted to present the TCE's intentions to some extent; they wrote several lines about the students' un-earthing of mathematics from climbing. However, some months after the project, none of them wrote anything that could be interpreted as *guided* un-earthing of angles in climbing; none of them mentioned anything regarding neither the teacher's role nor how the learning should take place.

During the period between day two and day three, Frode performed mathematical archaeology on the activities in his physical education lessons. But this led to a teaching that was interpreted as deductive meso space teaching, category C in Table 1. During the school year Frode's teaching practice changed: he and his students un-earthed mathematics from their running, and his students had even made a written report about this. This indicates that he had internalized and implemented some of the intentions of the TCE. However, there is no sign of inductive approaches in Frode's teaching. There is no claim regarding inductive work in the Norwegian mathematics curriculum (KD, 2006). This is opposed to the NCTM (2000) geometry standard which points out explicitly that the grade 6-8 students should create and critique inductive and deductive arguments.

The TCE findings lead to the hypothesis that teachers can be guided to re-invent a climbing approach to angles the following way: At first, a phase where the teacher experiences different approaches to teaching: deductive versus inductive, and meso space versus micro space. These constitute the four categories in table 1. One important point here may be a discussion about how to bridge the gap between the different categories. A short instructional DVD has been made as a basis for such a discussion (Fyhn, 2007). In addition the teachers need to discover the power of a mathematical archaeology approach by performing it themselves. After having experienced this phase, the teacher is ready for trying to perform a *guided un-earthing* of angles in climbing. One more instructional video has been made for this purpose (Fyhn, 2008). However, neither these videos nor these ideas have been researched.

Six months after the last climbing day, Therese was working with outdoor education at a non-degree granting college, and there she paid no attention to mathematics education. Frode was approaching the subject of geometry in his teaching schedule, and without explaining he underlined that his way of teaching differed from the researcher's. He claimed: "You must let the students perform activities that they enjoy. The challenge is to find the mathematics in these activities." He still was interpreted to mean that he was performing the mathematical archaeology on his students' activities, and then he explains the un-earthed mathematics via a deductive approach. The findings from the TCE indicate that future research and instructional design of this kind carefully need to give the teachers time for gathering experiences with and reflecting upon guided un-earthing opposed to other approaches to teaching.

## Notes

<sup>1</sup> To belay means to secure the climber with a breaking device connected to the rope in case the climber falls. The climber will then be hanging by the rope.

## Acknowledgements

Without the cooperative attitude of Therese and Frode throughout this project, this research could not have been fulfilled. I have spent lots of their time by returning to them in my

attempts to find out what they really meant. Odd Valdermo has asked many critical questions, and contributed to my validating of data throughout this project. Ragnar Soleng at the Faculty of Science also has given valuable support, while Linn Sollied Madsen has language checked the text.

## References

- Andersen, S.S. (2003). *Case-studier og generalisering. Forskningsstrategi og design*. (2rd. Ed.) Bergen, Norway: Fagbokforlaget.
- Berthelot, R. and Salin, M.H. (1998). The Role of Pupil's Spatial Knowledge in the Elementary Teaching of Geometry. In C. Mammana & V. Villani (Eds.) *Perspectives on the Teaching of Geometry for the 21<sup>st</sup> Century*. Dordrecht: Kluwer.
- Brekke, G., Kobberstad, T., Lie, S. and Turmo, A. (1998). *Hva i all verden kan elevene i matematikk? Oppgaver med resultater og kommentarer. TIMSS*. Oslo: Universitetsforlaget AS.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: D. Reidel.
- Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel.
- Freudenthal, H. (1991). *Revisiting Mathematics Education. China Lectures*. Dordrecht: Kluwer.
- Fyhn, A. (2001a). En matematikklek som er populær blant jenter. In *Norsk pedagogisk tidsskrift*, 5, 493-497
- Fyhn, A. (2001b). Matematikk innbakt i kunst og håndverk. In *Tangenten*, 4, 40-42.
- Fyhn, A. (2002a). Aktiv persepsjon og læring av geometri. In *Tangenten*, 2, 44-46.
- Fyhn, A. (2002b). (Ed.) *Snø-ski- geometri. Kompendium i matematikk og kroppsøving. Studenter og lærere ved Høgskolen i Tromsø*. Bergen: Caspar Forlag.
- Fyhn, A. (2006). A climbing girl's reflection about angles. In *Journal of Mathematical Behavior*, 25, 91 – 102.
- Fyhn, A. (2008). A climbing Class' Reinvention of Angles. In *Educational Studies in Mathematics*, 67 (1), 19-35.
- Fyhn, A. (2007). *Angles as Tool for Grasping Space: Teaching of Angles Based on Students' Experiences with Physical Activities and Body Movement. A Dissertation for the Degree of Philosophiae Doctor*. Tromsø, Norway: Department of Mathematics and Statistics, Faculty of Science, University of Tromsø
- Gjone, G. and Nortvedt, G. (2001). *Kartlegging av matematikkforståelse. Veiledning til geometri F og I*, Oslo: Directorate for Primary and Secondary Education.
- Gravemeijer, K. and Cobb, P. (2006). Design Research from a Learning Design Perspective, in J. van den Akker, K. Gravemeijer, S. McKenny and N. Nieveen (Eds.) *Educational Design research*, pp. 17-51. GB: Oxon: Ruthledge.
- Henderson, D. W. and Taimina, D. (2005). *Experiencing Geometry. Euclidean and non-Euclidean with history*. New York: Cornell University.
- Van den Heuvel-Panhuizen, M. (2005). Measurement and Geometry in Line, in M. van den Heuvel-Panhuizen and K. Buys (Eds.) *Young Children Learn Measurement and Geometry. A Learning-Teaching Trajectory with Intermediate Attainment Targets for the Lower Grades in Primary School. Tal Project*, pp. 9-14. NL: Utrecht: Freudenthal Institute (FI), Utrecht University and National Institute for Curriculum Development (SLO).
- Heuvel-Panhuizen, M. van den (2003). The Didactical Use of Models in Realistic Mathematics Education: An Example from a Longitudinal Trajectory on Percentage. In *Educational Studies in Mathematics*, 54, 9-35.



- Johnsen, V. (1996). Hva er en vinkel? In *NOMAD Nordisk Matematikdidaktikk/ Nordic Studies in Mathematics Education* 4, 25 - 49
- KD, Ministry of Education and Research (2006). *Knowledge promotion. Mathematics subject Curriculum..* Oslo: Utdanningsdirektoratet/ Norwegian Directorate for Education and Training. Retrieved March 22, 2010 from [http://www.udir.no/upload/larerplaner/Fastsatte\\_lareplaner\\_for\\_Kunnskapsloefet/english/Mathematics\\_subject\\_curriculum.rtf](http://www.udir.no/upload/larerplaner/Fastsatte_lareplaner_for_Kunnskapsloefet/english/Mathematics_subject_curriculum.rtf)
- Krainer, K. (1993). Powerful tasks: A contribution to a high level of acting and reflecting in mathematics instruction. In *Educational Studies in Mathematics*, 24, 65-93.
- KUD, Ministry of Church and Education (1987). *Mønsterplan for grunnskolen*. Oslo: Kirke- og undervisningsdepartementet og Aschehoug & Co.
- KUF, Ministry of Education and Research (1996a). *Læreplanverket for den 10-årige grunnskolen*. Oslo: Nasjonalt Læremiddelsenter.
- KUF, Ministry of Education and Research (1996b). *The curriculum for the 10- year compulsory school*. Retrieved March 22, 2010 from [http://www.udir.no/L97/L97\\_eng/index.html](http://www.udir.no/L97/L97_eng/index.html)
- Lakoff, G. and Núñez, R. (2000). *Where mathematics comes from. How the embodied mind brings mathematics into being*. New York: Basic Books.
- Leatham, K. (2006). Viewing Mathematics Teachers' Beliefs as Sensible Systems. In *Journal of Mathematics Teacher Education*, 9, 91-102.
- Lerman, S. (2002). Situating research on mathematics teachers' beliefs and on change. In G. C. Leder, E. Pehkonen and G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 233-243) Dordrecht: Kluwer.
- Mathisen, G. and Fyhn, A. (2001). Student-skidag i Bak-Olsen: Lærte matematikk via kroppssøving. In *Håja*, 2
- Mitchelmore, M. C. and White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. In *Educational Studies in Mathematics*, 41, 209 – 238.
- NCTM, National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics, Geometry Standard for Grades 6-8*. Retrieved March 22 from <http://standards.nctm.org/document/chapter6/geom.htm>
- Schoenfeld, A. H. (1998). Toward a theory of teaching-in.context. In *Issues in Education*, 4 (1)
- Skovsmose, O. (1994). *Towards a Philosophy of Critical Mathematics Education*. Dordrecht/Boston/London: Kluwer Academic Publishers.
- Torkildsen, O.E. (2006). *Mathematical Archaeology on Pupils' Mathematical Texts. Unearthing of Mathematical Structures. Dissertation for the Degree of Dr. Philos.* Oslo: University of Oslo, Faculty of Education
- Törner, G. Rolka, K., Rösken, B. and Sriraman, B. (2010). Understanding a Teacher's Actions in the Classroom by Applying Schoenfeld's Theory Teaching-In-Context: Reflecting on Goals and Beliefs. In B. Sriraman and L. English (Eds.) *Theories of Mathematics Education. Seeking New Frontiers* (pp.401-420). Heidelberg/Dordrecht/London/New York: Springer
- Wood, T. and Berry, B. (2003). Editorial. What does "Design Research" offer Mathematics Teacher Education? *Journal of Mathematics Teacher Education*, 6, 195-199.
-

## Vectors in Climbing

Anne Birgitte Fyhn<sup>1</sup>  
University of Tromsø, Norway

*Abstract: In this article, the work on mesospace embodiments of mathematics is further developed by exploring the teaching and learning of vector concepts through climbing activities. The relevance and connection between climbing and vector algebra notions is illustrated via embedded digital videos.*

**Keywords:** climbing and mathematics; digital media; egocentric vs allocentric representations; embodied mathematics; flow; mathematisation; mathematical archaeology; mathematics and physical education; meso space; meso space representations; reflective research practice; vectors; teaching and learning geometry

### Introduction

The teaching of vectors is not an easy task for mathematics teachers. It is difficult to illustrate the concepts of vector calculus exclusively by means of blackboard and chalk (Perjési, 2003), and it also is difficult to succeed in the teaching of vectors (Poynter & Tall, 2005). The literature on the teaching and learning of vectors is very scant in mathematics in comparison to physics education. The dearth of studies addressing student difficulties in vector concepts seems troubling given the basis it forms for vector Calculus, applied analysis and other areas of mathematics. Fyhn (2007, 2010) used children's experiences with physical activities and body movement as a basis for the teaching of angles. In this previous work the focus was on angles in a climbing context. One idea behind the work with angles in a climbing context for primary school students was to lay the groundwork for further work with vectors in upper secondary school (ibid.). The Norwegian curriculum introduces vectors as part of the mathematics syllabus in the second grade of upper secondary school.

A climbing video can introduce vectors without a blackboard and chalk, and such a video might aid in students' understanding. In addition, a video has the possibilities of freezing the picture and drawing vectors on it, in order to guide the watcher's focus. This paper presents and analyses the climbing-and-vectors-video "Vectors in Climbing" found at [http://ndla.no/nb/node/46170?from\\_fag=56](http://ndla.no/nb/node/46170?from_fag=56).

-Click link to activate video stream-

The actors in this video are two skilled climbers, Birgit and Eirik, who are second year students at upper secondary school. Both had chosen theoretical mathematics as a subject at

---

<sup>1</sup> Førsteamanuensis Matematikdidaktikk  
Lærerut. og pedagogikk  
Universitetet i Tromsø  
[anne.fyhn@uit.no](mailto:anne.fyhn@uit.no)

school. To research whether this video may be a useful contribution to the teaching of vectors, the following question was posed

How are vectors introduced in the video “Vectors in climbing”?

### **The choice of context**

Students should learn mathematics by developing and using mathematical concepts and tools in day-to-day situations which make sense to them (Van den Heuvel-Panhuizen, 2003). The fantasy world of fairytales as well as the formal world of mathematics might function as contexts for mathematical problems. The point is that the students have to experience the context as “real” in their own worlds of thoughts and imagination (ibid.).

Climbing is an activity that can be carried out by most young people given the opportunity and basic instruction. Several places in Norway have indoor climbing walls at a reasonable distance from the upper secondary schools. Many classes include students who climb regularly as a recreational activity. Students with some climbing experiences are familiar with the climbing context, and they will be able to imagine what goes on in the video. Those with no climbing experiences will to some extent be able to imagine what goes on. Most people have experienced climbing trees or rocks as children.

The concept of ‘flow’ was introduced by Csikszentmihalyi (2000). *Flow* is the holistic feeling one gets when one acts with complete involvement. Humans seek *flow* for its own sake, and not as a means for achieving something else. Professional athletes have referred to the notion of flow during moments of peak performance and described it as a feeling that transcends duality.

Achievement of a goal is important to mark one’s performance but is not in itself satisfying. What keeps one going is the experience of acting outside the parameters of worry and boredom: the experience of flow. (ibid. s. 38)

One quality of *flow* experiences is that they reveal clear and unambiguous feedback to a person’s acting. Climbing, chess and mathematics are examples of *flow* activities (ibid.). The climbing video offers vectors as an integrated part of an activity which is experienced as exciting and fun by some young people. In the video Birgit and Eirik claim, “... in climbing you always have to ... have fun!”

### **Bodily geometry<sup>2</sup>**

Human mathematics is based upon experiences with our bodies in the world, and therefore the only mathematics we are able to know is the mathematics that our bodies and brains allow us to know (Lakoff & Núñez, 2000). When someone presents you with an idea, you need the appropriate brain mechanism to be in place for you to (hopefully) understand it, and then learn it or reject it. Consequently, “mathematics is fundamentally a human enterprise arising from basic human activities” (ibid. p. 351). This perspective matches Freudenthal (1973), who claimed that geometry is grasping space. “Space” here means the space in which the child lives, breathes and moves. The goal of teaching of geometry is that the students are able to live, breath and move better in the space (ibid.).

---

<sup>2</sup> The author uses the term “bodily geometry” to refer to experiences that involve the entire body as opposed to focusing on specific parts or motions as done by those working in semiotics.

Bodily geometry is geometry teaching that builds upon the students' own experiences with their complete bodies present in a three dimensional world. The goal of bodily geometry is a) that the students are able to understand the outcomes of their own bodily actions, and b) that the students are able to use geometry as a tool for living and moving better in particular contexts. The two central goals of the video are i) that students are able to use climbing as a resource for their understanding of vectors, and ii) that students may use their knowledge about vectors to analyse what they do and then improve some of their movements. The last goal mainly concerns students who are climbers.

### **Different conceptions of space**

According to Lakoff and Núñez (2000) space is conceptualized in two different ways through the history of mankind. We cannot avoid the first one, the 'naturally continuous space'; "It arises because we have a body and a brain and we function in the everyday world. It is unconscious and automatic" (ibid., s. 265). Climbing takes place in this *naturally continuous space* where coordinates and axes do not exist. Descartes' metaphor, 'numbers-as-points-on-a-line' lead to the metaphor 'space-as-a-set-of-points', which is ubiquitous in contemporary mathematics. Even though it takes special training to think in terms of the *set-of-points* metaphor, it is taken for granted throughout contemporary mathematics (ibid.): One aim in the Norwegian mathematics syllabus LK-06 was "to visualize vectors in the plane, both geometrically as arrows and analytically in co-ordinate form." (KD, 2006, p. 3), and one more aim was "to calculate and analyze lengths and angles to determine the parallelity and orthogonality by combining arithmetical rules for vectors" (ibid.). The video concerns vectors in the *naturally continuous space* and no calculations take place. A large part of the video focuses on how vectors may be added. A basic understanding of this is necessary before the students are able to succeed in vector calculus.

Berthelot and Salin (1998) subdivided space into three different categories with respect to size; microspace which corresponds to grasping relations, mesospace which corresponds to spatial experiences from daily life situations, and macrospace which corresponds to the mountains, the unknown city and rural spaces. Poynter and Tall (2005a; 2005b) used the term physical activity if the students move an object they hold in their hand, even if they are sitting on a chair performing a micro space activity. This is not bodily geometry; bodily geometry takes place in meso space. The vector video presents bodily geometry, because the actors move their own bodies in meso space.

The mathematical abstraction 'line segments' are connected with 'long objects' via the phenomenon of the rigid body. And a rigid body remains congruent with itself when displaced (Freudenthal, 1983). A climber is familiar with her or his own limbs and body. One point of climbing is not to fall off the wall; how to place the body and how to make it rigid. Freudenthal further claimed

I am pretty sure that rigidity is experienced at an earlier stage of development than length and that length and invariance of length are constituted from rigidity rather than the other way around. Rigidity is a property that covers all dimensions while length requires objects where one dimension is privileged or stressed. (ibid., p. 13)

In bodily geometry a rigid body is an important part of 'length'; a long arm is able to reach longer than a short arm. And if you stand on your toes you are able to reach a little further.

### **Climbing: a natural alternation between *egocentric* and *allocentric* frames of reference**

Berthoz (2000) referred to ‘personal space’, ‘extra personal space’ and ‘far space’, where *personal space* in principle is located within the limits of a person’s own body. According to Berthoz (ibid.) the brain uses two different frames of reference for representing the position of objects. The relationships between objects in a room can be encoded either ‘egocentrically’, by relating everything to yourself, or an ‘allocentric’ way, related to a frame of reference that is external to your body. Only primates and humans are genuinely capable of *allocentric* encoding. Children first relate space to their own bodies and the ability of *allocentric* encoding appears later (ibid.). Moreover, “*allocentric* encoding is constant with respect to a person’s own movement; so it is well suited to internal mental simulation of displacements” (ibid., p. 100).

When you are trying to ascend a passage of a climbing route, you encode the actual passage *egocentrically* within your *personal space*. But when you stand below a climbing route considering whether or how to ascend it, you exercise *allocentric* encoding in *extra personal space* by considering how the route’s different elements and your body relate to each other.

The climber Fredrik explains how this takes place

[http://www.uvett.uit.no/iplu/fyhn/fredrik/fredrik\\_forklarer.html](http://www.uvett.uit.no/iplu/fyhn/fredrik/fredrik_forklarer.html)

-Click link to activate video stream-

Climbing can offer students good possibilities for moving back and forth between *egocentric* and *allocentric* representations. While one climber is struggling with a problem, other fellow climbers are often keen on trying to solve the problem themselves. Through their *allocentric* encoding they may imagine how to solve the problem. Next, they make use of *egocentric* coding in their attempts to solve the problem.

### **Mathematical archaeology**

One idea of the video was to start with an activity which many students find exciting and fascinating, and then search for some of the mathematics within the activity. This is done through mathematical archaeology (Fyhn, 2010). A mathematical archaeology is an educational activity where mathematics is recognized and named. This involves being aware that some activities are in fact mathematics (Skovsmose, 1994). “An aim of a mathematical archaeology is to make explicit the actual use of mathematics hidden in the social structures and routines” (ibid., p. 95). The term ‘archaeology’ refers to a systematically ‘un-earting’ (Torkildsen, 2006) of something hidden. In this case vectors are *un-earthed* from three climbing situations. In Fyhn (2010), a detailed description of how angles concepts were un-earthed was given. This idea is now extended to the realm of vectors.

The idea was to offer a more thorough presentation of vectors in one particular context, instead of making a presentation of all aspects of vectors. So the intention was not to start with the learning goals in the syllabus, and then search for some teaching that will fulfill these demands. Climbing is an activity, which requires full concentration. Every move concerns problem solving; how do you position your body so that your hands and feet do not lose the grip of the holds? Climbing does not necessarily concern bodily geometry, because most climbers do not reflect on any mathematics while climbing. By performing mathematical archaeology on some climbing situations, bodily geometry can be the focus.

Vectors might be a useful tool for explaining what goes on in some climbing situations, but it is not given that these situations will constitute a proper basis for the teaching of vectors. This is one limitation of this approach.

### **Intuitive and formal understanding**

Fischbein (1994) claimed, that mathematics should be considered from two points of view; as formal deductive knowledge as found in high-level textbooks, or as a human activity. He pointed out that the ideal of mathematics, as a logically structured body of knowledge, does not exclude the necessity to consider mathematics as a creative process. Mathematics is a human activity, which is invented by human beings.

Fischbein (ibid.) considered the interactions between three basic components of this human activity: 1) The formal aspect: Axioms, definitions, theorems and proofs, 2) The algorithmic component: We need skills and not only understanding, and skills can be acquired only by practical, systematic training, 3) Intuition: Intuitive cognition, intuitive understanding, intuitive solution. Intuitive cognitions may sometimes be in accordance with logically justifiable truths, but sometimes they may contradict them. The point is that “it is the intuitive interpretation based on a primitive, limited, but strongly rooted individual experience that annihilates the formal control or the requirements of the algorithmic solution, and thus distorts or even blocks a correct mathematical reaction” (ibid., p. 244). A climbing video may offer a connection between formal and intuitive understanding of vectors, as long as the students have an expectation of what will happen in the video; a connection between the paragraphs 1 and 3 above.

### **The teaching of vectors**

In Norway vectors is not part of the compulsory mathematics syllabus. The students, who choose a theoretical perspective on the mathematics subject, are introduced to vectors in their second year at upper secondary school. Those who choose to learn physics, meet vectors there too, later in the same school year. The national syllabus (KD, 2006) included the following competence aims about vectors for the second year of upper secondary school mathematics

The aims of the studies are to enable students to ...

- visualize vectors in the plane, both geometrically as arrows and analytically in co-ordinate form
- calculate and analyze lengths and angles to determine the parallelity and orthogonality by combining arithmetical rules for vectors (ibid., p. 3)

No research report is found on the teaching of vectors in Norwegian schools. Actually, there are not many research reports to be found regarding the teaching of vectors where computer software is not used. One reason for this might be Perjési's (2003) claim, that it is difficult to illustrate vector calculus by blackboard and chalk. Poynter and Tall (2005a; 2005b) however, represent an exception; they challenged the accepted British ways of teaching.

The British teaching culture uses practical approaches to practical problems (Poynter & Tall, 2005a); vectors are introduced in practical situations in physics. But students find the teaching difficult. The mathematics teachers are aware of their students' problems, but the tendency is that they focus on what the students do incorrectly, without saying why (Poynter & Tall, 2005b). A possible solution to this problem is: One important goal of the teaching is that “the parallelogram law, the triangle law and the addition of components of vectors are all seen as different aspects of the same concept” (ibid., p. 132). This goal is not included in the

Norwegian syllabus. Poynter and Tall (ibid.) pointed out that significant distinctions of meaning in different contexts affect the way students think; they apply the triangle law for displacements and the parallelogram law for forces. Their suggested way to reach the above goal was that the students first construct the essential meaning of ‘free vector’ and then apply this meaning to vectors in different contexts. The overall goal is “to create conceptual knowledge with a relational understanding of the concepts, rather than a procedural knowledge with an instrumental understanding of separate techniques” (ibid., p. 133). In the vector case, a relational understanding of concepts is interpreted to mean a) relations between different aspects of vectors and b) relations between vectors and other concepts. Other relevant concepts are magnitude, direction and angle.

A conceptual knowledge with a relational understanding of vectors can constitute a proper basis for students on their way to achieve the competence aims in the Norwegian syllabus. The Norwegian syllabus to a large extent left it up to the teachers to find out how the students may achieve the syllabus’ aims.

### **Method**

The video “Vectors in climbing” presents vectors in three climbing situations. The actors are two young skilled climbers who were also highly successful in mathematics at school. The climbing situations were analysed with respect to vectors, and the result is bodily geometry, which concern vectors. Therefore the video was categorised as mathematical archaeology (Skovsmose, 1994) with regards to the climbing situations. Some apparently hidden vectors in climbing were uncovered by being identified, named and described. The archaeology was carried out by an un-earthing (Torkildsen, 2006) of vectors that are hidden in these climbing situations. The analyses will enlighten how vectors were presented throughout the video.

The analyses in this paper was performed with respect to Poynter and Tall’s (2005b) goal; a conceptual knowledge with a relational understanding of the vector concept. This means that a) the triangle method, the parallelogram method and the decomposition of vectors should be treated as three aspects of the vector concept, and that b) relations between vectors and other concepts are in focus. Furthermore, the video was analysed with respect to c) the treatment of *free vector*, due to Poynter and Tall’s (2005b) treatment of this aspect. The two actors’ responses to the video was also analysed with respect to connections between intuitive and formal understanding. After they had watched an early version of the video, Birgit and Eirik were interviewed and asked their opinion about the relevance of the video for teaching as well as for their own climbing. The Data in this study is the video “Vectors in climbing” and the researcher’s hand written notes of Birgit and Eirik’s answers to the interview questions.

### **Relevance**

As stated previously, vectors are introduced to the Norwegian students who choose to study theoretical mathematics in upper secondary school’s second grade. Typically students claim that vectors are strange, abstract and difficult, and have nothing to do with the mathematics they know. Birgit from the video explained that she met vectors in a meteorology context at school, and that was difficult because she knew nothing about meteorology. Whether a student considers a syllabus subject important or not, depends on whether the student feels that it concerns her or his overall life situation (Mellin-Olsen, 1987). For the teacher this means to focus on the following questions, “...how does the lesson relate to her students’ conception of the important totalities of their world, and how can this lesson eventually

transform this totality?” (ibid., s. 33). Consequently the choice of context and teaching method is important in the introduction of new subjects in mathematics. The video intended to relate to what climbing youths consider as important totalities in their world.

A demonstration video like “Vectors in climbing” does not have the same value as it could have if the students were able to perform the climbing themselves. One advantage of the video is that students might watch and re-watch it at their own speed. According to Valdermo (1989) the teaching advantage from a demonstration may increase if it is made close to the intentions of a student experiment. The benefit of a demonstration is that the teacher is given a possibility of steering and controlling what goes on.

### **Mathematics and physics**

In a climbing context vectors means ‘forces’. And forces are part of the physics syllabus for Norwegian upper secondary school students. However, mathematics and physics are related disciplines.

...we see them rather as related disciplines that attempt to mathematise and physicalise our surrounding world, i.e. to describe phenomena in physical and mathematical terms in order to act and deal with them in a sensible way. (Doorman, 2005, p.3)

To ensure the quality of the physics part of the climbing video, two professionals were consulted: one retired physics teacher and one physicist. The teacher has 90 ECTS credits in physics, a master’s degree in meteorology, and thirty years’ experiences as an upper secondary physics teacher, but no climbing experiences. The physicist was a skilled and experienced climber with a range of experiences from the Yosemite Valley to Norwegian crags and local indoor walls. He held a Ph.D. in physics.

### **Analyses of the video**

The video presents descriptive mathematics; there are no tasks with numbers for calculations. The last part, focus five, includes some problem solving tasks. Skovsmose (1994) indicated that teachers prefer applied mathematics to descriptive mathematics. Descriptive mathematics might contend no tasks, and that could be the reason why many teachers prefer to work with applied mathematics. There is a risk that some teachers (and maybe students, too) dislike the video because most of it presents descriptive mathematics. So when the video is tested, it is necessary to be aware about this.

Poynter and Tall (2005b) underlined that one teaching goal is to create conceptual knowledge with a relational understanding of the concepts, rather than a procedural knowledge which includes no more than an instrumental understanding of separate techniques. It is of great importance that the “triangle law”, the “parallelogram law” and the addition of components of vectors all are seen as different aspects of the same concept. In the climbing video these three aspects are closely connected. But the video does not focus on what a vector is, except for claiming that a vector has a magnitude and a direction. Poynter & Tall (ibid.) described one possible way of guiding the students towards grasping a *free vector* concept in order to enlighten what a vector is. For this purpose they start out with an activity where the students physically displace some figures. The video’s weakness is its treatment of the very basic part of vector teaching. While Poynter and Tall (ibid.) described an inductive approach, the video is trapped by a deductive approach: repeated definitions.



The video's strength is that it underlines relations between different aspects of vectors and relations between vectors and angles. The triangular method and the parallelogram method are presented in two of the situations, while decomposition of vectors is presented in all three situations. Relations between vectors and scalars are briefly presented in the introduction, where a picture of a thermometer is assisted by a short text: "A magnitude without direction is called a scalar. It does not make sense to speak of the direction of a temperature". But the video does not mention that the length of a vector also is a scalar.

Relations between vectors and angles are presented in all three situations, and in addition, this part includes several problems posed to the audience. This indicates a need for a profound knowledge about the direction of vectors. According to Fyhn (2007) angle is a central element in the knowledge of vectors. Mitchelmore and White (2000) claimed, that the easiest angles to grasp are those where both of the legs are visible. In a climbing context there are a variety of angles where both of the legs are visible. There are situations where only one leg is visible, too. When the climbing-and-vectors video focuses on one particular angle, the picture is frozen and the angle is drawn on the picture, in order to a) underline which angle is in focus and b) make both of the legs visible. The video also challenge the students in imagining the connection between a change in angle and a change in what happens in the video. This is done in order to focus on the effect of different actions, just as Poynter and Tall (2005b) pointed out.

Vectors having the same length and same direction are called equivalent (Anton, 1981). According to Poynter and Tall (2005b), the *free vector* as a manipulable concept correlates with success on a delayed test about vectors and "The problem is how to encourage the students to construct the flexible concept of *free vector*" (ibid., p. 133). This is a central element in the work with vectors. The point with a transformation is not the transformation itself, but its effect

The *effect* of a physical action is not an abstract concept. It can be *seen* and *felt* in an embodied sense. The idea is that, if students had such an embodied sense of the effect of a translation, then they could begin to think of representing it in terms of an arrow with given magnitude and direction. (ibid. p. 133)

Common examples of climbers' experiences with the use of their own bodies are: the effect of what happens, when a climber falls and the rope is tightened, or how a climber may place her or his body so that one of her or his feet can stand on a vertical surface. Climbers have a variety of experiences with *egocentric* and *allocentric* encoding of situations like these, while non-climbers to some extent might imagine them. The intention of the climbing video is to offer the students a possibility to imagine and to feel what happens through the different sequences, by an alternation between *egocentric* and *allocentric* encoding of *meso space* activities.

### **Birgit and Eirik's responses to the video**

The first version of the video was shown separately to Birgit and Eirik. Eirik expressed that he grasped most of it. Birgit watched the video in silence, and said that some of it was still difficult. Some other young climbers, who occasionally were present, claimed that what followed after focus one, was theoretical and difficult to understand. This indicates that the video probably does not reach those who do not understand what a vector is. The video presents vectors in situations that the students may imagine. But on this occasion, the deductive approach to what a vector is, confirmed these young climbers' beliefs that mathematics was too difficult to them to grasp.

Birgit and Eirik were interviewed separately immediately after they had watched this version of the video. Concerning the video's relevance, Birgit claimed

If I had watched this video when I was working with vectors at school, I certainly would have been thinking more about it in my climbing. When I worked with vectors at school, I needed quite a long time to grasp what a vector really is and how to use it in practical situations. If I had seen the video and had taken part in the production of it while I was working with vectors at school, it probably would have been easier for me to understand.

We learned a lot about vectors through meteorology, but I do not know anything about meteorology. But if you learn about vectors through more ordinary things, then you might connect it to things you already know.

To Birgit, climbing is an ordinary thing. Her claims are interpreted to mean that she is not able to imagine what meteorology is, and consequently it is difficult for her to grasp what a vector is. Van den Heuvel-Panhuizen (2003) pointed out that it is important to present the mathematics in contexts that the students are able to imagine, just to avoid experiences like the one described by Birgit.

Birgit and Eirik were also asked which of the sequences they preferred, and they were also given the opportunity to suggest other possible sequences that might be relevant for the video. They both liked the *foot work* best, Birgit because she thinks a lot about it in her climbing. In addition she believes this sequence is the easiest one to grasp. Eirik said, regarding *foot work*, "This is something that I do all the time. This sequence is directly transferable to climbing, so this one was most useful to my climbing... I may use mathematics as a tool to find out how to do things the easiest way." In addition Eirik argued why he believed that all three sequences were good. Both Birgit and Eirik are interpreted to mean that they believe *foot work* to be the sequence with most relevance to climbing. Both are interpreted that a) their intuitive and their formal understanding (Fischbein, 1994) of vectors were connected and b) their *allocentric* coding of the sequence is used in an *egocentric* coding (Berthoz, 2000) of their own climbing. Regarding *pendulum*, Eirik claimed, "The circular arc in relation to the fall - I did not think about that before. So, purely mathematically, this sequence was the most thought-provoking to me." Birgit, on the other hand, was interpreted to feel that this sequence presented too much at a time: "I would like to have watched *pendulum* some more times". Her answer may also be interpreted to mean that this use of video is useful, because the watcher may stop the video and watch each sequence a desired number of times.

Birgit claimed that the video was long enough with three sequences. But she suggested that there might have been some more sequences related to the climbing of overhangs, but that this maybe could turn out to be difficult to present in a video. Birgit's suggestion will be left as an idea to teachers who want to let their students do project work concerning vectors. Eirik had one definite suggestion: "One thing I remember from school was that many students struggled with addition and subtraction of vectors. Consequently, that issue might be included." Eirik's comment here was of great importance because addition of vectors was included in the video at this moment. His answer was interpreted to mean that the video treated this issue too superficially. Birgit and Eirik responded differently to the question about whether climbing might be useful for the understanding of climbing. Birgit doubted it:

I don't know. If you start explaining by vectors it probably becomes confusing. You have to know these things, even if you don't know that it is vectors. You have to know where the rope is oscillating in order to have a safe fall. And every climber knows that you must not stand far from the wall while belaying.

Eirik was more positive:

... when I watched *foot work*, I thought that I might use this elsewhere as well. It is difficult to say anything definite about it, because it goes in and out of my consciousness. Earlier I performed things because I knew it would become easier. But now I reason about which vector that makes it easier.

Their different answers to this question may be interpreted to mean that Eirik has a better understanding of vectors than Birgit, and that he is able to see more possibilities. He shows a clear analytic relation to vectors when it comes to climbing. Birgit's point is that you can climb at a high level without knowing anything about vectors. The response from the other young climbers who watched the video together with Birgit, indicate that the video does not represent a good introduction to vectors. The video presents relations between different aspects of vectors, but that is of little help to those who do not know what a vector is! Some weeks after the interview, Birgit and Eirik showed the video to some classes in lower secondary school. These students found vectors difficult. Some days later Eirik claimed that he believed no one but Birgit and him would learn anything from the video. This claim is interesting, and it supports the idea of using the video as an example of how to perform mathematical archaeology. Students might use the video as a basis for interdisciplinary project work in mathematics and physical education. The students may then work in groups according to what physical activity they prefer to enlighten.

### Responses from Birgit and Eirik's School

The video was presented to Birgit and Eirik's mathematics class at school, after they had completed their work with vectors that year. Their mathematics teacher was present in addition to the school's headmaster and the researcher. When the video ended, the class applauded spontaneously, and their comments were positive. Maybe they acted positively just because they got some attention, or maybe they acted positively because they could relax and watch some of their classmates. The first comment was that the examples were understandable. In the following school year some of the teachers planned to use the video as part of their mathematics teaching. Due to some technical difficulties the video could not be used there. But one of the teachers showed it as part of the physics teaching for Eirik's class. Some weeks before these students' final exam at upper secondary school, the researcher met one boy from this class by chance. The boy asked "You are the one who made the vector video, aren't you?" The question was verified, and followed by a return question about whether he thought the video was of any use. The boy answered that he did not find the video particularly useful before he started working with forces as part of the physics subject. And when he was preparing for his final physics exam, he found the video both interesting and informative. This supports the results from the analyses, that the video has a missing link somewhere between focus one and focus two; the video does not provide the students with an introduction to addition of vectors.

### Summing up

A search for the word 'vector' in the mathematics education research literature gives few hits except for those where the context is 'computer based environments'. One reason for this might be Perjési's (2003) claim, that it is difficult to illustrate the concepts of vector calculus exclusively by means of blackboard and chalk. The video 'Climbing and Vectors' meets this challenge because it is based on *bodily geometry*, and it presents *allocentric encoding* (Berthoz, 2000) of activities that the students can easily imagine. The activities take place in the *naturally continuous meso space*, and the students are able to imagine what goes on in the video as van den Heuvel-Panhuizen (2003) pointed out. However, the analyses of the video points out an important weaknesses with the video: a lack of a fundamental explanation of

what a vector is. The young climbers, who watched the video together with Birgit, claimed the video was abstract, most likely because of the deductive approach to what a vector is. One way of doing this, is as Poynter and Tall (2005b) described; through *free vector*. The video has some strengths; the close relations between a) the presentation of connections between the triangular method, the parallelogram method and additions of components of vectors, b) the demonstration of relations between vectors and angles. The video also offers a presentation of how to perform mathematical archaeology upon one particular context.

One important question remains. How will the video “Vectors in climbing” influence the students’ understanding of vectors? Students with no climbing experiences will probably respond differently from the skilled climbers. The mathematical archaeology in the video was performed by the researcher and two climbers; it is far from obvious how other climbers will respond to the video. Climbers and other young active people may use the video as background for how to perform mathematical archaeology on different activities. The video has two goals: i) that students are able to use climbing as a resource for their understanding of vectors, and ii) that students may use their knowledge about vectors to analyse what they do and then improve some of their movements. Eirik gives examples of things he did not think of before watching the video. He also claims that he has started using vectors as a tool for analyzing his climbing. He is interpreted to make use of *allocentric* encoding (Berthoz, 2000) in these analyses. His answers concern both these goals. Birgit answered that the footwork sequence concerned most of what she does in climbing. She may be interpreted to mean that the video suits students with some climbing experience. Mellin-Olsen (1987) pointed out that the teaching should relate to what the students experience as their world, and Birgit supports this

It is always nice if you are familiar with things without knowing that it is a vector. If you learn about vectors through climbing, and you know a lot about climbing, then it is easier... If you learn about vectors through daily things, then you might connect it to some of your previous knowledge.

### Acknowledgements

First of all I need to thank the climbers Birgit and Eirik. Without them and their attitude towards the video project, this research would not have taken place. Thanks to the physicists Bjørn Braathen and Knut Fyhn for valuable comments to the video. And thanks to the climber Fredrik who let me make the video clip about *egocentric* and *allocentric* encoding. Thanks to Odd Valdermo for his constructive critiques and thanks to the reviewers in Acta Didactica Norway for their respond to a part of this text in Norwegian. Yngvar Natland has given lots of support in editing the videos.

### References

- Anton, H. (1981). *Elementary linear algebra*. Third edition. NY: John Wiley & sons
- Berthelot, R. & Salin, M.H. (1998). The Role of Pupil’s Spatial Knowledge in the Elementary Teaching of Geometry, i C. Mammana & V. Villani (Eds.) *Perspectives on the Teaching of Geometry for the 21st Century*, pp. 71- 78. Dordrecht: Kluwer Academic Publishers.
- Berthoz, A. (2000). *The Brain’s Sense of Movement*. Harvard University Press, Cambridge, Massachusetts. Original title: *Le Sens du Mouvement*, 1997.
- Csikszentmihalyi, M. (2000). *Beyond Boredom and Anxiety. Experiencing Flow in Work and Play*, Jossey-Bass, San Francisco. First published 1975.

- Doorman, L.M. (2005). *Modelling motion: from trace graphs to instantaneous change*. NL: Utrecht: CD-β Press, Center for Science and Mathematics Education. Also published as a dissertation at Utrecht University.
- Fischbein, E. (1994). The interaction between the formal, the algorithmic, and the intuitive components in a mathematical activity, in R. Biehler, R.W. Scholz, R. Strässer & B. Winkelmann (Eds.) *Didactics of Mathematics as a Scientific Discipline*, pp. 231-245. Dordrecht: Kluwer Academic Publishers.
- Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: D. Reidel.
- Fyhn, A.B (2007). *Angles as Tool for Grasping Space: Teaching of Angles Based on Students' Experiences with Physical Activities and Body Movement. A Dissertation for the Degree of Philosophiae Doctor*. University of Tromsø: Department of statistics and mathematics, Faculty of science.
- Fyhn, A.B (2010). Climbing and Angles: A Study of how two teachers internalise and implement the intentions of a teaching experiment. *The Montana Mathematics Enthusiast*, 7(2&3), 275-294.
- KD, Ministry of Education and Research (2006). Mathematics for the natural sciences – programme subject in programmes for specialization in general studies, in *Knowledge promotion*, retrieved December 10th 2009 from [http://www.utdanningsdirektoratet.no/upload/larerplaner/Fastsatte\\_lareplaner\\_for\\_Kunnskapsloftet/english/Natural\\_science\\_mathematics/Mathematics\\_for\\_the\\_natural\\_sciences\\_rtf](http://www.utdanningsdirektoratet.no/upload/larerplaner/Fastsatte_lareplaner_for_Kunnskapsloftet/english/Natural_science_mathematics/Mathematics_for_the_natural_sciences_rtf)
- Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from. How the embodied mind brings mathematics into being*. New York: Basic Books.
- Mellin-Olsen, S. (1987). *The Politics of Mathematics Education*. US: MA: Kluwer Academic Publishers.
- Mitchelmore, M. C. & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation, in *Educational Studies in Mathematics*, 41, 209 – 238.
- Perjési, I. H. (2003), Application of CAS for teaching of integral-transforming theorems, in *ZDM, The International Journal on Mathematics Education* 35 (2), 43 - 47
- Poynter, A. & Tall, D. (2005a). Relating Theories to Practice in the Teaching of Mathematics. Paper at *The Fourth Congress of the European Society for Research in Mathematics Education, CERME 4*, Spain.
- Poynter, A. & Tall, D. (2005b). What do mathematics and physics teachers think that students will find difficult? A challenge to accepted practices of teaching, in D. Hewitt & A. Noyes (Eds.) *Proceedings of the sixth British Congress of Mathematics Education*, University of Warwick, pp. 128-135. Retrieved January 11th 2009 from <http://www.bsrlm.org.uk/IPs/ip25-1/BSRLM-IP-25-1-17.pdf>
- Skovsmose, O. (1994). *Towards a Philosophy of Critical Mathematics Education*. Dordrecht/Boston/London: Kluwer Academic Publishers.
- Torkildsen, O.E. (2006). *Mathematical Archaeology on Pupils' Mathematical Texts. Unearthing of Mathematical Structures. Dissertation for the Degree of Dr. Philos.* Oslo: University of Oslo, Faculty of Education
- Valdermo, O. (1989). Demonstrasjoner i naturfagundervisningen, in A. Kvam, I. O. Størkersen & O. Valdermo (Eds.) *Metodisk veiledning i naturfag*, Oslo: Gyldendal norsk forlag A/S and Rådet for videregående opplæring.
- Van den Heuvel-Panhuizen, M. (2003). The Didactical Use of Models in Realistic Mathematics Education: An Example from a Longitudinal Trajectory on Percentage, in *Educational Studies in Mathematics* 54, 9-35.

## **Rules Without Reason: Allowing Students to Rethink Previous Conceptions**

*Mary Mueller<sup>1</sup>*  
*Seton Hall University*  
*South Orange, NJ*

*Dina Yankelewitz*  
*The Richard Stockton College of New Jersey*  
*Pomona, NJ*

*Carolyn Maher*  
*Rutgers University*  
*New Brunswick, NJ*

*Abstract: This paper reports on the strategies chosen by a group of sixth-grade students in an urban informal learning program as they worked to solve an open-ended, non-routine task. In particular, the paper focuses on the ability of these students to rise above their previous, procedure-based misconceptions and arrive at a mathematically reasonable solution. Aspects of the problem task, the problem-solving environment, and, importantly, of the nature of the teacher's interventions are analyzed to determine the conditions that encouraged students to approach mathematics as a logical, meaningful, sense-making activity.*

**Keywords:** Cuisenaire™ rods ; conceptual versus procedural; fractions; meta-cognition; open-ended tasks; problem solving; procedural learning; sense making; student reflections; teacher interventions

### **Introduction**

The goal of building students' mathematical reasoning and their ability to create and defend proofs is a priority among mathematicians, mathematics educators, and policy leaders. Researchers concur that reasoning and proof are the foundation of elementary

---

<sup>1</sup> Mary Mueller  
400 South Orange Avenue  
South Orange, NJ 07059  
908-458-3909  
[muellmf@shu.edu](mailto:muellmf@shu.edu)

and middle-grades mathematics learning and are necessary for building sustainable mathematical understanding (Hanna, 2000; Hanna & Jahnke, 1996; Polya, 1981; Stylianides, 2007). Ball and Bass (2003) contend that “the notion of mathematical understanding is meaningless without a serious emphasis on reasoning” (p. 28). They argue that understanding is founded on reasoning in that students must use reasoning to understand relationships and make connections to new ideas.

Unfortunately, students struggle with reasoning, especially reasoning involving analysis of relationships between quantities, rather than just reasoning of numbers or operations alone (Kouba & Wearne, 2000). They also experience difficulty when explaining and justifying their thinking (Arbaugh, Brown, Lynch, and McGraw, 2004). Although students may be able to solve complex problems, they are not always able to cognitively defend their solutions, explain, or justify the process they used to reach an answer. This struggle is compounded by the fact that teachers often focus primarily on addressing the learning of “mathematics facts and concepts” and on “learning skills and procedures needed to solve routine problems” (Silver, Alacaci, & Stylianou, 2000, p. 339).

In addition, students’ knowledge that is not fully developed or not understood often hinders their ability to reason. When ideas that are not fully understood or developed are used in arguments, they can become obstacles that encumber correct reasoning. In the process of making sense of a problem situation or creating an argument to support a solution, conflicts between incomplete knowledge and new knowledge schemes emerge. In these cases, students need to be open to adapt their newly developed knowledge to accommodate their new understandings. This can come about in situations where students can talk about and explain their reasoning as well as hear the explanations of other students (Wearne & Kouba, 2000, p.189). Encouraging communication in problem solving can promote students’ growth in mathematical understanding and in solving more complex problems. With support from others, students can extend their knowledge, sometimes by building alternative representations, and by sharing, discussing and arguing about their ideas.

In this paper, we share an episode from an after-school mathematics program where a group of sixth-grade students were prompted to rethink what they know about fraction “rules”. They did this by building their own evidence and convincing themselves and others to believe in their power to reason.

## **Theoretical Framework**

### **Understanding**

Robert B. Davis (1992) explains that, given opportunities, students will create their own ways of understanding and build representations and understanding based on their previous knowledge and experiences. However, Davis points out that what students learn is built upon this foundation of understanding and that future learning may also be limited by previous understanding. Davis suggests that understanding is achieved when one is able to fit a new idea into a larger structure of earlier constructed ideas. Davis’ (1992) theory of understanding notes, “One gets a feeling of understanding when a new idea can be fitted into a larger framework of previously-assembled ideas” (p. 228). Davis refers to the representational structures that a learner builds as a collection of *assimilation*

*paradigms*. Davis and Maher (1993) describe these assimilation paradigms as the act of fitting new ideas into larger frameworks of previously assembled ideas. The learner views the new idea as “just like” or “similar to” an existing experience and uses this to accommodate the new knowledge (Davis & Maher, 1993). Davis (1992) notes the existence of obstacles to understanding, which he refers to as “cognitive obstacles” and describes as “improperly chosen assimilation paradigms that lead to incorrect ways of thinking or that are limited in their scope” (p. 226).

Often, the mathematical instruction in schools does not connect to children’s natural, experience-based understandings. Instead, it requires students to adapt their reasoning styles to fit those valued by schools (Malloy, 1999), which may result in the aggregation of cognitive obstacles. Too often, the traditional approach to teaching mathematical concepts emphasizes students’ memorization of rules and procedures and manipulation of symbols. Many of these rules are meaningless to children, having been learned by rote methods (Davis 1994). Erlwanger (1973) reports the case of Benny, a twelve-year-old boy in the sixth grade using Individually Prescribed Instruction (IPI), who believed that mathematics consists of different rules for different problems that were invented at one time but work like magic. In Benny’s eyes, mathematics was not a rational and logical subject where one has to reason, analyze, seek relationships, make generalizations, and verify answers; rather, it was a game where one discovers the rules and uses them to solve problems (Erlwanger, 1973). Benny created his own “rules” for adding fractions based on what he perceived as random procedures. Kamii and Diminck (1998) argue that teaching rules and conventions can be harmful because they cause children to relinquish their own ideas and disconnect the content from the concepts. When exposed to this kind of instruction, students often remember erroneous rules and procedures, as was seen with Benny. Kamii and Warrington (1999) propose that the focus of instruction should shift from teaching that emphasizes physical and social knowledge to that which values and encourages children’s own reasoning.

Yackel and Hanna (2003) concur and argue the view of mathematics as reasoning can be contrasted with the view of mathematics as a rule-oriented activity. Other researchers support the fact that the sole teaching of algorithms can be detrimental and counterproductive to the development of children’s numerical reasoning. Mack (1990) came to this conclusion after finding that algorithms often keep students from even trying to use their own reasoning. Through her work with eight sixth-grade students, she found that while all eight students began with serious misconceptions about fractions, they also had a considerable repertoire of informal knowledge that allowed them to solve real-life fraction problems. Mack found that students built on informal knowledge when given the opportunity to connect problems represented symbolically to real-life situations that they understood. However, faulty, isolated knowledge of rote procedures often got in the way of problem solving, since students often remembered erroneous algorithms and had more faith in these rules than in their own thinking. Overall, Mack (1990) concluded that students could use informal knowledge about fractions to build meaning and understanding. She noted that the knowledge of rote procedures can interfere with the building of fractional understanding and that the results give evidence to the argument that students should be taught concepts before procedures.

According to Skemp (1971), “...to understand something is to assimilate it into an appropriate schema” (p. 45). As a result, a student’s level of understanding depends upon



the schema he or she has created during instruction. Understanding develops as students form connections between new and old knowledge and create appropriate schemas to make sense of new knowledge (Davis & Maher, 1997). These schemas are built on previous understanding as students make connections between schemas. Often, students run into roadblocks that they must overcome by building alternative representations and sharing ideas. Students exhibit logical understanding when they are afforded the opportunity to justify their reasoning in a community of learners and are able to adapt previous [mis]understandings/beliefs. The learning environment is a crucial component of the successful development of reasoning, and in turn, it is necessary for the development of understanding. According to Martino and Maher (1999), if they are given a supportive environment, students will appreciate the value in discussing their ideas with classmates in order to help formulate and refine their own thinking.

### Communities of Learners

Vygotsky (1978) hypothesized that students internalize the discussions that occur in group contexts. Cobb, Wood, and Yackel (1992) built on this claim by stressing the role of social interaction in the construction of mathematical knowledge and calling attention to the role of discussions in a mathematical community. Cobb (2000) stresses the importance of studying students' mathematical activity in the social environment of the classroom. He states that students' mathematical understanding is based on the relationship between individual student activity and overall classroom practice. The classroom micro-culture has a significant influence on the meaning students make of mathematics, as well as their use of explanations.

According to Maher and Davis (1995), a learner is influenced by the representations created by others in his or her community. They explain that "[d]ifferent forms of reasoning coexist within this community and different forms of representation are used to transmit ideas, explore and extend these ideas, and then act upon them" (Maher & Davis, 1995, p. 88). Maher and Davis use a ribbon and bow metaphor to describe this community of learners, explaining, "A single ribbon may be viewed as a particular path pursued by a particular child in developing an idea; the bows, varied in complexity and size, might be viewed as ideas received from other members of the community and acted upon by individual children in such a way as to influence their development" (p. 88). A strong community of learners where ideas are transferred from one student to another exemplifies the bow metaphor.

According to Maher and Martino (1996, 2000), learners bring personal experience to every mathematical investigation. Given a context that includes a mixture of personal exploration and social interaction, coupled with students' mental representations, knowledge and beliefs may be refined and modified to build new ideas and theories. The responsibility shifts from the teacher to the learner, and therefore the learner has the opportunity to make sense of the mathematics through careful reasoning and the building of arguments (Maher, 2005).

The classroom micro-culture or community in which learning occurs shapes the meaning that students construct. A community of learners is created through mathematical discussions in a rich social environment. Within this community, ideas are shared, modified and agreed upon and students' understanding is built. Collaboration is often viewed as the support provided by learners within a community in the form of

missing pieces of information needed to solve the problem (Francisco & Maher, 2005). Francisco and Maher contend that a form of collaborative work that involves group members relying on each other to generate, challenge, refine, and pursue new ideas is also important (p. 369). With this type of collaboration, rather than piecing together their individual knowledge, the students work together to build new ideas and ways of thinking.

### **Methodology and Data Sources**

#### **Participants**

This research is a component of a larger ongoing longitudinal study, Informal Mathematics Learning Project, (IML)<sup>2</sup> that was conducted as part of an after-school partnership between a state university and a school district located in an economically depressed, urban area. The district's student population consists of 98 percent African American and Latino students. Our study focuses on the development of reasoning of middle-school students. In the IML program the participants met twice a week over a six week period in four cycles that each focused on different task strands (combinatorics, probability, algebra, or fractions). During each session, students were invited to work collaboratively on open-ended mathematical tasks. Task choice played an important role in the project's objectives, since if the tasks were too simple, the students' schemas would not be enhanced, but if they were too difficult, the students would be discouraged from engaging in an attempt to find solutions. We report on the first cohort of students, 24 sixth-graders, who, over five, 60-75 minute sessions worked on fraction tasks, interacted with peers, and had ample time to explore, discuss and explain their ideas. The sessions were facilitated by two of the researchers (university professors) who designed the program. For the purpose of this paper these two researchers will be referred to as teachers. The students worked in heterogeneous groups of four and participated in whole class discussions. In each session, problems were posed and students were asked to explore solutions in their small groups first and then to share with the whole class. They were invited to collaborate and discuss their ideas with one another, were encouraged to justify and make sense of their solutions, and were challenged to convince one another of the validity of their reasoning.

#### **The Tasks and Tools**

The strand of tasks was developed from an earlier research intervention with fourth-grade students. The fourth grade students had not been previously introduced to fractions. Through their engagement in the fraction tasks, it was documented that the students used reasoning to compare fractions, find equivalent fractions and perform operations on fractions after working on the tasks (Steencken & Maher, 2003). The current study differed in that the sixth grade students had already been introduced to fraction rules and procedures with little conceptual understanding of meaning. We therefore chose to give the sixth-grade students tasks that focused on rational numbers and the use of manipulative materials to build concrete models.

---

<sup>2</sup> This work was supported in part by grant REC0309062 (directed by Carolyn A. Maher, Arthur Powell and Keith Weber) from the National Science Foundation. The opinions expressed are not necessarily those of the sponsoring agency and no endorsements should be inferred.

The students were given Cuisenaire™ rods for building models to represent their solutions. A set of Cuisenaire rods, as shown in Figure 1, contains 10 colored wooden or plastic rods that increase in length by increments of one centimeter.

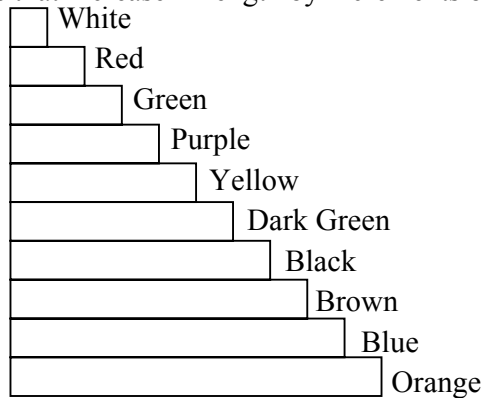


Figure 1. “Staircase” Model of Cuisenaire Rods.

As part of introducing students to working with the rods, the researchers explained that the rods are given permanent color names. These names, along with their respective lengths, are: white (1 cm); red (2cm); light green (3 cm); purple (4 cm); yellow (5 cm); dark green (6 cm); black (7 cm); brown (8 cm); blue (9 cm); and orange (10 cm).

The rods were assigned variable number names and students were challenged to identify fractional relationships with respect to a specified given unit. For example, one task was: “*What number name would you give to the dark green rod if the light green rod is called one? Discuss the answer with your group*” (Maher, 2002). Students were then provided time to investigate their solutions and make claims in their small groups. They were encouraged to use the rods to build models as justifications in support of their solutions. After being afforded ample time to build models, collaborate, and justify their solutions, groups were invited to the overhead projector to share their findings with the class. During these whole group discussions, students shared their findings, challenged each other, and often reflected on and revised their solutions.

#### Data Collection and Analysis

Each session was videotaped with four different camera views. The cameras focused on different groups of students and one of the cameras also captured the presentations at the overhead projector. Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The video data were described at frequent intervals; critical events (episodes of reasoning) were identified and transcribed, and codes were developed to flag for solutions offered by students and the justifications given to support these solutions. Arguments and justifications were coded according to the form of reasoning being used, direct or indirect, and as valid or invalid, based on whether or not the argument started with appropriate premises and the deductions within the argument were a valid consequence of previous assertions. Students’ construction of solutions and their subsequent justifications were then traced across the data in an effort to document and analyze their journey to mathematical understanding.

## **Results**

Throughout the intervention, the teachers were careful to refrain from correcting students' errors, rather allowing the students to discuss their ideas until their classmates either refuted or agreed with their solutions and justifications. Upon analysis of the data, an interesting pattern was noticed. As the sessions progressed, students were observed to confidently share their solutions, both correct and incorrect, and other students often challenged these solutions and countered with refutations of the justifications. In particular, episodes during which a student provided an invalid solution were likely to evidence rich student interaction, varied forms of reasoning, and the emergence of a soundly-justified, correct solution.

An episode from the fourth session of the after-school program is used to illustrate the ways in which students overcame previous misunderstandings while attending to open-ended tasks. In the previous session, the blue rod was given the number name one and students were asked to find a number name for the white rod and the red rod. During session four, students initiated naming the remaining rods in the set (when the blue rod was named one). Students first worked in small groups and then shared results with the larger community. An excerpt from each forum is provided below.

### **A Group of Three: Chanel, Dante and Michael**

Chanel lined up five white rods next to the yellow rod and used direct reasoning to name the yellow rod five-ninths. She then initiated the task of naming all of the rods using the staircase model (see Figure 1). She named the remainder of the rods, using direct reasoning based on the incremental increase of one white rod or one-ninth and used the staircase model as a guide and named the rods until she arrived at the orange rod. As she was working she said the names of all of the rods, "One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths, ten...-wow, oh, I gotta think about that one, nine-tenths". Chanel showed Dante her strategy of using the staircase to name the rods and explained the dilemma of naming the orange rod to Dante, "See this is one-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths - what's this one [the orange rod]?". Dante replied, "That would be ten-ninths. Actually that should be one. That would start the new one". He initially named the orange rod ten-ninths but corrected himself and said that the orange rod would "start a new cycle"; and named it one-tenth. Michael named the orange rod a whole and explained that it was equivalent to ten white rods and Chanel agreed.

Chanel: It should be called a whole.

Dante: This is one, this is nine-ninths also known as one. This should be blue and this would start the new one – would be one-tenth.

After students worked for about five minutes drawing rod models, Dante told the group that that he heard another group calling the orange rod ten-ninths.

Dante: Why are they calling it ten-ninths and [it] ends at ninths?  
Michael: Not the orange one. The orange one's a whole.  
Dante: But I'm hearing from the other group from over here, they calling it ten-ninths.  
Michael: Don't listen to them! The orange one is a whole because it takes ten of these to make one.  
Dante: I'm hearing it because they speaking out loud. They're calling it ten-ninths  
Michael: They might be wrong! ...  
Chanel: Let me tell you something, how can they call it ten-ninths if the denominator is smaller than the numerator?  
Dante: Yeah, how is the numerator bigger than the denominator? It ends at the denominator and starts a new one. See you making me lose my brain.

As the students were working a teacher joined their group. Dante shared his conjecture, "It's the end of it and it starts the new one to one-tenth because the blue ends it and so the orange starts a new one just like - pretend there were smaller ones than just a white. So this would be considered like blue, a one". The teacher reminded him that the white rod was named one-ninth and that this fact could not change. Again she asked him for the name of the orange rod and he stated, "It would probably be ten-ninths". When prompted, Dante explained that the length of ten white rods was equivalent to the length of an orange rod. The teacher asked Dante to persuade his partners.

Chanel: No, because I don't believe you because –  
Michael: I thought it was a whole.  
Dante: But how can the numerator be bigger than the denominator?  
R1: It can. It is. This is an example of where the numerator is bigger than the denominator.  
Chanel: But the numerator can't be bigger than the denominator, I thought.  
Michael: That's the law of facts.  
R1: Who told you that?  
Chanel: My teacher.  
Dante: One of our teachers  
Michael: That's the law of math.

In the above dialogue, we see that even though Dante named the orange rod ten-ninths, using previous knowledge (of the name of the white rod) and a concrete model, he still questioned his answer. His prior understanding of the "rule" was so strong that he questioned himself even after building a concrete model and explaining the concept.

### The Whole Class

At the end of the session groups were asked to share their results with the class.. Malika and Lorrin named the orange rod ten-ninths and reported that they initially thought the numerator could not be larger than the denominator.

Lorrin: Because, before, we thought that because we knew that the numerator would be larger than the denominator and we thought that the denominator always had to be larger but we found out that that was not true. Because two yellow rods equal five-ninths, and five-ninths plus five-ninths equal ten-ninths.

Kia-Lyn and Kori explained that the orange rod had two number names.

Kia-Lynn: We found that ... the orange one has two number names. So because the orange one and the blue one – I thought that – our group had found out - that the orange is bigger than the blue one but when you add a one-ninth, a white rod, to the blue top it kind of matches. It kind of matches and we found out that you can also call the orange rod one and one-ninth.

Kori: So we were saying that if this [blue] is called one –

Kia-Lynn: It's also called one – um ten-ninths as Malika and Lorrin had said. But if you have one...white rod and you add it to the blue, it's one-ninth plus one is one and one-ninth and so if the blue rod and one white [they are using overhead rods to show a train of the blue rod and a white rod lined up next to an orange rod as shown in figure 2]. If you put them together then this means that it's ten-ninths also known as one and one-ninth

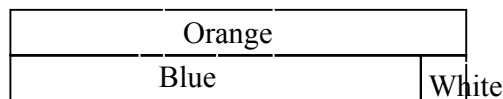


Figure 2. Kia-Lynn's model for naming the orange rod one and one-ninth.

Finally, Dante presented his strategy:

Dante: Well all I did was start from the beginning – start from the white – and you and all the way to the orange and what – like Kia-Lynn's group just said - I had found a different way to do it. Because all I - I had used an orange, two purples, and a red and since these two are purple and this is supposed to be purple but I had purple and I used a red since four and four are eight so which will make it eight-ninths right here and then plus two to make it ten-ninths. [He builds the model shown in figure 3 on the OH] That's what I made.

R2: So it's another way of showing that orange is equivalent to ten-ninths?

Dante: Um hum. And then I just did it in order – then the one I did right here – I just did it in order of whites by doing ten whites.

Orange		
Purple	Purple	Red

Figure 3. Dante's model for naming the orange rod ten-ninths

### Discussion and Implications

Specific factors in the after-school session enabled the students to challenge and revise their ways of thinking about mathematics. These factors included, but were not limited to, the following: challenging, open-ended tasks that invite students to extend their learning as they create and justify solutions, the opportunity to build models using concrete materials, the promotion of student collaboration in small groups and the opportunity to share ideas in the whole class forum, strategic teacher questioning, and the portrayal of student as determinant of what makes sense.

The tasks were open-ended such that students could expand on a given task, as in the above episode when the students initiated naming all of the rods in the set. In addition, the nature of the tasks and the time allotted for exploration allowed students to work at their own pace and readiness level. As the other students grappled with convincing themselves that ten-ninths was a viable number name, Kia-Lyn and Kori took the task to the next level and showed that ten-ninths is equivalent to one and one-ninth. In an environment that allowed students to act as teachers and present new concepts to their peers, the students were exposed to new ideas in a manner that was conducive to their assimilation.

In addition, the students were provided tools to build models and therefore they could conceptualize the fraction relationships. The Cuisenaire rods offered a concrete, visual model of ten-ninths and the students were thereby provided the means to show, using concrete evidence that this fraction did indeed exist. At the end of the session, Kori and Kia-Lynn presented a concrete model to show that these two fractions were equivalent and further justified this solution. These experiences offered the class new schema upon which to build and make connections (Davis & Maher, 1997) and the students were empowered to use these ideas and apply them to similar mathematical problems. Students, afforded the opportunity to explore, become convinced of, and justify their revised ways of thinking, became owners of the mathematical ideas they created. This ownership gave them the confidence to build new ideas and relationships.

The physical environment promoted student collaboration and it was further encouraged by the researchers asking students to listen to each other's ideas and to judge the merit of each other's justifications. Importantly, the researcher's careful questioning prompted students to explain their reasoning and invite their classmates to evaluate their thinking. When Chanel first grappled with naming the orange rod the teacher suggested she share her dilemma with Dante. After being afforded more time to think about the

task, Dante was asked to explain his thinking. Rather than correcting Dante, the teacher reminded him of the facts that were already established (that the white rod was named one-ninth). This subtle prompt enabled him to revise his thinking through the use of his own reasoning. Dante was then asked by the teacher to explain his thinking and convince his partners that his reasoning was correct. By working to convince his partners, Dante was able to reaffirm his reasoning and further convince himself of its validity.

Dante was again encouraged to have confidence in his own thinking during the second phase of the activity. After students were provided the opportunity to explain their thinking and discuss their ideas in their small groups, they participated in a whole class discussion, providing the opportunity for them to validate their ways of reasoning about the problem. Further, the arguments presented by others introduced them to alternative models and justifications. Although Dante used the staircase model to incrementally increase the names of the rods by one-ninth as he worked with his partners, he chose a different representation during his whole class presentation. After viewing the other presenters and listening to their presentations, his thinking was validated and thus he expressed confidence in his solution. This confidence led him to show two alternative models for naming the orange rod.

Malika and Lorrin shared that they, too, had previously believed that the numerator of a fraction could not be larger than the denominator; however, their reasoning and the concrete evidence that they used to show that five-ninths plus five-ninths is equivalent to ten-ninths was a stronger influence on their ultimate decision. In an environment that encourages reasoning, these students learned to trust their own logical ability and were thereby able to challenge and rethink their earlier understanding.

Overall, the teacher encouraged the students to revise their thinking and overcome their misconceptions. This was done via open-ended tasks, concrete materials, physical environment, and strategic questioning. More importantly, we can learn from what the teacher did *not* do during this session. The teacher did not correct the students and tell them that the rule that they had erroneously recalled was incorrect. Instead, she presented them with facts and allowed Dante to convince his group of the truth of his argument. Instead of replacing one teacher-initiated rule with another, the students were invited to resolve their differences and together agree on the mathematical validity of their solutions.

The teacher allowed the students to replace the previously learned rule by giving them the opportunity to independently build their own schemas for improper fractions. Instead of telling the children that ten-ninths is equivalent to one and one-ninth and showing them the proper algorithm to make this conversion, the teacher allowed the students to discover this on their own, in a setting that encouraged discussion and argumentation, and one that guaranteed students' safety as they offered their ideas for critique by their peers.. When students are encouraged to reason and thereby become members of a community of engaged, active learners, they are able to exhibit understanding and build trust in their own thinking. As students learn to rely on their sense making and reasoning skills, they become better situated to build new, meaningful knowledge. Thus students learn that errors in thinking are not bad; rather, they are the stepping stones to increased understanding. The findings from this analysis suggest that by allowing students to work together in solving mathematical tasks, with minimal teacher assistance or intervention, and in an environment that provides the security



needed to participate in open discussions of their ideas, regardless of their mathematical validity, students can learn to reason, can improve their understanding of basic concepts, and can be empowered to exercise agency over their own knowledge of mathematics.

## References

- Arbaugh, F., Brown, C., Lynch, K., & McGraw, R. (2004). Students' ability to construct responses (1992-2000): Findings from short and extended constructed-response items. In P. Kloosterman & F.K. Lester (Ed.s), *Results and Interpretations of the 1990-2000 Mathematics Assessments of the National Assessment of Educational Progress* (pp. 337-363). Reston, VA: NCTM.
- Ball, D.L., & Bass, H. (2003). Making mathematics reasonable in school. In J.Kilpatrick, W.G.Martin, and D. Schifter (Eds.), *A Research companion to the Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: NCTM.
- Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A.E. Kristin & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Erlbaum.
- Cobb, P., Wood, T., & Yackel, E. (1992). Learning and interaction in classroom situations. *Educational Studies in Mathematics*, 23, 99-122.
- Davis, R.B. (1992). Understanding "Understanding". *Journal of Mathematical Behavior*, 11, 225-241.
- Davis, R.B. (1994). What Mathematics Should Students Learn? *Journal of Mathematical Behavior*, 13, 3-33.
- Davis, R. B. & Maher, C. A. (1993). What are the issues? In R. B. Davis & C. A. Maher (Eds.), *Schools, Mathematics, and the World of Reality* (pp. 9-34). Boston: Allyn and Bacon.
- Davis, R. B. & Maher, C. A. (1997). How students think: The role of representations. In L. D.English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 93-115), Hillsdale, NJ: Lawrence Erlbaum Associates.
- Erlwanger, S.H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26.
- Francisco, J.M. & Maher, C.A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, 24, 361-372.
- Hanna, G (1991). Mathematical Proof. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Kluwer, Dordrecht.
- Hanna, G. (2000). A critical examination of three factors in the decline of proof. *Interchange*, 31(1), 21-33.
- Hanna, G., & Jahnke, H. N. (1996). Proof and Proving. In Bishop, A., et al. (Eds.), *International Handbook of Mathematics Education*, Dordrecht: Kluwer Academic Publishers.

- Kamii, C. & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L. J. Morrow & M. J. Kenney (Eds), *The teaching and learning of algorithms in school mathematics: 1998 yearbook*. (pp.130-140). Reston, VA: The National Council of Teachers of Mathematics.
- Kamii, C. & Warrington, M.A. (1999). Teaching fractions: Fostering children's own reasoning. In L.V. Stiff & F.R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12 NCTM 1999 Yearbook* (pp82-92). Reston, VA: National Council of Teachers of Mathematics.
- Kouba, V.L. & Wearne, D. (2000). Whole number properties and operations. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 141-163). Reston, VA: NCTM.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal Knowledge. *Journal for Research in Mathematics Education*, 21, 16-32.
- Maher, C.A. (2002). How students structure their own investigations and educate us: What we have learned from a fourteen year study. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the International Group for the Psychology of Mathematics Education (PME26)* (Vol. 1, pp.31-46). Norwich, England:
- Maher, C.A. (2005). How students structure their investigations and learn mathematics: Insights from a long-term study. *Journal of Mathematical Behavior*, 24(1), 1-14.
- Maher, C.A. & Davis, R.B. (1995). Children's exploration leading to proof. Mathematical Science Institute of London, University of London, 87-105.
- Maher, C.A. & Martino, A.M. (1996). The development of the idea of mathematical proof: a 5-year study. In F. Lester (Ed.), *Journal for Research in Mathematics Education*, 27(2), 194-214.
- Maher, C.A. & Martino, A.M. (2000). From patterns to theories: Conditions for conceptual change. *The Journal of Mathematical Behavior*, 19, 247-271.
- Malloy, C.E. (1999). Developing mathematical reasoning in the middle grades: Recognizing diversity. In L.V. Stiff & F.R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12 NCTM 1999 Yearbook* (pp.82-92). Reston, VA: National Council of Teachers of Mathematics.
- Martino, A.M. & Maher, C.A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior*, 18(1), 53-78.
- Polya, G. (1981). Mathematical discovery. New York: Wiley.
- Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, 22(4), 405-435.
- Silver, E.A., Alacaci, C. & Stylianou, D.A. (2000). Students' performance of extended constructed-response tasks. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 307-343). Reston, VA: NCTM.
- Skemp, R. R. (1971). *The Psychology of Learning Mathematics*. Penguin Books.

- Steencken, E.P. & Maher, C.A. (2003). Tracing fourth graders' learning of fractions: early episodes from a year-long teaching experiment. *The Journal of Mathematical Behavior* 22(2), pp. 113-132.
- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38, 289-321.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Translated by Michael Cole, V. John-Steiner, S. Scribner, E. Souberman. Harvard University Press, Cambridge, MA.
- Wearne, D. & Kouba, V.L. (2000). Rational numbers. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 163-193). Reston, VA: NCTM.
- Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin. & D. Schifter. (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp 227-236). Reston, VA: NCTM.

## A Semiotic Reflection on the Didactics of the Chain Rule<sup>1</sup>

Omar Hernández Rodríguez<sup>2</sup> and Jorge M. López Fernández<sup>3</sup>  
Graduate School of Education, University of Puerto Rico at Río Piedras

**Abstract.** According to (Fried, 2008), there is an intrinsic tension in trying to apply the history of mathematics to its didactics. Besides the widespread feeling that the introduction of didactic elements taken from the history of mathematics can detract the pedagogy of mathematics from the attainment of important goals, (Fried, 2008, p. 193) describes a pair of specific pitfalls that can arise in implementing such historical applications in mathematics education. The description in (Fried, 2008), is presented in the parlance of Sausserian Semiotics and identifies two semiotic “deformations” that arise when one fails to observe that the pairing between signs and meanings in a given synchronic “cross-section” associated with the development of mathematics need not hold for another synchronic cross section at a different time. In this exposition, an example related to an application of the history of the chain rule to the didactics of calculus is presented. Our example illustrates the semiotic deformations alluded by (Fried, 2008), and points out a possible explanation of how this may lead to unrealistic pedagogical expectations for student performance. Finally, an argument is presented for the creation of a framework for a historical heuristics for mathematics education, possibly beyond the bounds of semiotics.

**Keywords:** chain rule; composition of functions; differentiation; historical heuristics; history of analysis; history of mathematics; Sausserian semiotics

### 1. Application of Sausserian Semiotics to Mathematics

The problem of applying the history of mathematics to mathematics education and its relation to semiotics is broadly discussed by (Fried, 2008). Specific examples are given of distortions that arise in the application of semiotics to the history of mathematics when failure to distinguish differences between synchronic and diachronic descriptions of the body of mathematics

---

<sup>1</sup> The authors are especially grateful to Robert L. Knighten, Adjunct Professor of Mathematics at Portland State University, for his help in clearing up some of the historical points treated in this article and related to the development of the chain rule as a statement of differentiability of the composition of two functions.

<sup>2</sup> E-mail address: omar.hernandez4@upr.edu

<sup>3</sup> E-mail address: lopez.jorgem@gmail.com

occurs. We shall not dwell on the elements of semiotics discussed there; in fact, the presentation in (Fried, 2008) depicts adequately the elements of semiotics relevant to the reading of this article, and also presents some of the details of the development of Sausserian Semiotics and its adequacy for framing the problem of applying the history of mathematics to mathematics education. Also, (Fried, 2008) dwells in a general fashion on some of the main ideas of semiotics, the contributions of Peirce to this field of knowledge, and also presents some examples related to the distortions that can arise in failing to differentiate between synchronic descriptions of the relations between signs and meanings (both in linguistics and mathematics) that occur at different time frames. In this article, we employ the framework put forth by Fried (2008) to discuss the application of the history of mathematics to the teaching of calculus, specifically to the didactics of the chain rule for the differentiation of the composition of two differentiable functions. It will be argued that failure to make the alluded distinction between diachronically distinct synchronic descriptions of the body of mathematics can result in unrealistic expectations regarding student understanding of the chain rule.

For the purposes of facilitating the exposition that follows, we review some of the descriptions presented by Fried (2008, p. 193) regarding two distortions that can arise when one fails to recognize the fact that the relations between signs and meanings in the history of mathematics, can be vastly different when diachronic differences between time periods are taken into account. The distortions, according to (Fried, 2008, pp. 193) are twofold. The first distortion consists of supposing that the synchronic relations between signs and meanings in a given historic period coincide with those thought to be the corresponding relations between the homologous signs and meanings of the present time (when mathematics education occurs). This distortion constitutes, in fact, the worst error a historian can make, that is, the error of

anachronism. The error involves contriving non existent or false synchronic relationships between signs and meanings in the given historic period. The second distortion described by Fried, and related to the failure to recognize diachronic differences, is the fabrication of false inferences regarding the evolution of signs, meanings and their pairings throughout the history of mathematics. In (Fried, 2008) two examples of these distortions are given, one in linguistics, due to Saussure (1974) himself, and another one related to the notion of function in Euler's times (Fried, 2008, p. 194).

In this note we discuss some of the signs and meanings associated with the notions of derivative and composition of functions, as related to the chain rule<sup>4</sup>. The issue here is the history of the chain rule since the publication of L'Hospital *Analyse des Infiniment Petits pour la Intelligence des Lignes Courbes* in 1696. Succinctly stated, the modern statement of the chain rule is taken to be one that relates the derivative of the composition of two functions with the individual derivatives of the functions composed (provided, of course, certain conditions are satisfied). Since the idea of composition of functions seems to have appeared in the literature at least a century after the publication of *Analyse des infiniment petits*<sup>5</sup>, it is impossible that the signs and meanings relevant to the statement of the chain rule in the seventeenth century are the same as those associated with the present version of the chain rule.

## 2. History of the Chain Rule

We now present a brief relation of the evolution of the mathematical ideas and relations that have come to be known as the "chain rule". The present day statement of the Chain Rule is a rather sophisticated one and presupposes the confluence, and consolidation of many mathematical ideas. In fact the modern statement of the Chain Rule is the following:

---

<sup>4</sup> that is, to the rule for differentiating the composition of two differentiable functions.

<sup>5</sup> As far as we can tell, the first "modern" version of the chain rule appears in Lagrange's 1797 *Théorie des fonctions analytiques*, (Lagrange, J. L., 1797, §31, pp. 29); it also appears in Cauchy's 1823 *Résumé des Leçons données à L'École Royale Polytechnique sur Le Calcul Infinitesimal*, (Cauchy, A. L., 1899, Troisième Leçon, pp. 25).

**Theorem 1.**

*If  $g$  is differentiable at  $c$ , and  $f$  is differentiable at  $g(c)$ , then  $f \circ g$  is differentiable at  $c$  and*

$$(f \circ g)'(c) = (f' \circ g)(c) \cdot g'(c).$$

The possibility of the succinct and beautiful statement contained in Theorem 1 presupposes a great deal of evolution of the underlying mathematical ideas and a commensurable amount of “negotiations” related to the corresponding signs and meanings. Here, the functions of the statement are assumed to be defined on neighborhoods of their points of differentiability, and the corresponding limits for the difference quotients are supposed to exist as real numbers. Furthermore, once the “correct” definition of the derivative for Euclidean spaces was discovered, the chain rule was extended to state a relation about the differentiation of composite functions on Euclidean spaces, thus changing very little the formal statement of Theorem 1; see (Dieudonné, 1960).

In (L’Hospital, 1696, p. 2), the difference of a variable  $y$  depending on an independent variable  $x$  is defined as the infinitesimal increment in  $y$  when  $x$  changes by an infinitesimal amount  $dx \neq 0$ . In modern notation that needs little explanation:  $dy = [y(x + dx) - y(x)]dx$ . In fact, in *Analysis des infiniment petits* (L’Hospital, 1696), curves are considered as polygons of an infinite number of sides of infinitesimally small lengths, so that if we were to extend the “side” of the curve  $y(x)$  that joins the points  $(x, y(x))$  and  $(x + dx, y(x + dx))$  in the graph of  $y$  as a function of  $x$ , we would, in fact, obtain the tangent line to the curve at  $(x, y)$ , whose slope is, no more and no less, than the quotient  $dy/dx$ . It should be noted that L’Hospital (1696) used a geometric argument that employs the similarity of infinitesimal triangles to show that the value of the desired slope is infinitely close to the indicated value  $dy/dx$ . If one follows mathematical convention and writes  $y'(x)$  for the quotient  $dy/dx$  (and this is, indeed, a quotient!) then, the

relation  $dy = y'(x)dx$  holds for all infinitesimals  $dx$ . In fact, in *Analyse des infiniment petits* the calculus of derivatives is really the calculus of “differences” of variables.

It may come as a surprise to the reader that nowhere in *Analyse des infiniment petits*, (L'Hospital, 1696), is the chain rule stated explicitly. This mystery is rather significant in more than one way. First, if we have differentiable variables  $y$  depending on  $u$  and  $u$ , in turn, depending on  $x$ , then  $dy = y'(u)du$  and  $du = u'(x)dx$  are the basic relations for the differences at the appropriate points, so that  $dy = y'(u)du = y'(u)u'(x)dx$ . From this, again, it is clear that  $dy = y'(u)u'(x)dx$ , and this is the chain rule.

Furthermore, this is true whether  $dx$  is zero or not. It may be even more surprising to realize that the statement of the chain rule is also absent in all of Euler's analysis books, *Introductio in analysin infinitorum*, (Euler, 1748, Vol. 1), (Euler, 1748, Vol. 2), and *Institutiones calculi differentialis*, (Euler, 1755). Furthermore, Euler did define the notion of a function in (Euler, L., 1748, Vol. 1), but he never treated the topic of the composition of functions in any of his writings, (Euler, 1748, Vol. 1), (Euler, 1748, Vol. 2) and (Euler, 1755).

As far as we can tell, the first mention of the Chain Rule<sup>6</sup> in the literature of calculus seems to be due to Leibniz (Child, 2007, p. 126), and it appears in a 1676 memoir (with various mistakes) in which he calculated  $d\sqrt{a+bz+cz^2}$  by means of the substitution  $x = a + bz + cz^2$ . In *Analyse des infiniment petits* (L'Hospital, 1696, pp. 3-4), the rules for calculating the differences of the basic algebraic combination of (differentiable) variables are given. L'Hospital posed the problem of calculating the difference of  $x^r$  for any “perfect or imperfect” power  $r$  (that is, for any rational power  $r$ ) and he answers his question by proving that  $dx^r = rx^{r-1}dx$ . In keeping with the style of *Analyse des infiniment petits*, after proving general rules, L'Hospital gave instances of

---

<sup>6</sup> as a rule for finding differences of expressions by means of substitutions.



the application of the rule to specific examples. In this case, the first example of the general rule  $dx^r = rx^{r-1}dx$  given by L'Hospital is the calculation of the difference  $d(ay - x^3)^3$ . The calculation is, as expected, a direct application of the chain rule and requires no expansion of the cube. No comment is made by L'Hospital to the effect that the application of the general rule (differentiation of the cube) to more complicated expressions necessitates an application of a special rule (the chain rule) whose statement or demonstration is nowhere to be found in his work (Campistrans, Lopez, and Rizo, 2009).

In our view, the example provided illustrates dramatically that the anachronisms that ensue from failing to understand the diachronic differences between the mathematics of different times can betray the existence of pitfalls and ill practices in the didactics of mathematics. Informal experiments performed in an introductory non standard calculus course at the University of Puerto Rico have shown that students have significant difficulties in identifying the composed functions before they are able to correctly apply the chain rule. On the other hand, the level of understanding of the chain rule improves when the algorithm is presented as differentiation after a substitution of variables. To the distant observer this may seem to be a trivial difference, but the history of mathematics shows that the notion of composition somehow requires a higher level of abstraction for its understanding. Similar remarks apply to the understanding of the chain rule by students when it is presented in nonstandard analysis parlance as contrasted with the usual standard analysis presentation, which requires arguments often seemed as much to do about nothing. Perhaps, it should be remarked as a sobering thought, that even if all the diachronic and synchronic semiotic deformations in the history of mathematics can be avoided, there will still remain what in our view is the most interesting part of the history of mathematics (and, also, the

part most related to mathematics education), and that is the inferences that can be made from it regarding optimal strategies for classroom teaching.

### **3. Towards a Historical Heuristics for Mathematics Education**

After (Toeplitz, 1963), it has been amply regarded that the so called “genetic approach” to mathematics education has special advantages. Toeplitz (1963) carefully points out the difference between history in general, as a compilation of facts, and the history of mathematics in particular, as a source of ideas for teaching mathematics. He remarks: “It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts and proofs” (Toeplitz, 1963, p. xi). In view of the semiotic considerations of this work, we venture to suggest the need of a sort of historic heuristics for mathematics education, in the vein, perhaps, of the heuristics of (Polya G., 1945) for problem solving, but which attend to the pairings the human mind makes between signs and meanings for the purpose of advancing mathematics knowledge. In our opinion, in the case of the chain rule, a strong argument can be made for the cognitive advantages of defining the derivative as a difference arising from an infinitesimal change, just like in (L’Hospital, 1696). To this, in our view, we owe the absence of explanations and the familiar and informal handling of the chain rule in (L’Hospital, 1696).

Kitcher (1983, p. 229) presents some compelling arguments for what we consider to be the cognitive advantages of what can be called “Newton’s kinematic metaphors” (thinking of fluents and fluxions as positions and velocities, respectively, of moving objects; see (Kitcher, 1983, p. 232)), and the appropriateness of infinitesimals as a cognitive vehicle for the “initial calculus<sup>7</sup>”, striving to describe mathematically the idea of “change”. In fact, in spite of all objections to the unclarities of the initial calculus, as voiced (mainly) by Bishop George Berkeley (Wilkins, 2002), the calculus was rapidly accepted and it developed in an unprecedented way. In the words of

---

<sup>7</sup> the calculus developed by Newton Leibniz, the Bernoullis, L’Hospital and Euler

(Kitcher, 1983, p. 230): “To understand how the power of the methods introduced by Newton and Leibniz outweighed the unclarities which attended them, we must begin with the problems which interested the mathematicians of the early seventeenth century.”; and further ahead on the same page: “Both Newton and Leibniz introduced new language, new reasonings, new statements and new questions into mathematics. Some of the new expressions were not well understood and the workings of some of the new reasonings were highly obscure. Despite of these defects, the changes they proposed were accepted quite quickly by the mathematical community, and the acceptance was eminently reasonable.” Thus, in spite of all logical difficulties, the methods of the initial calculus were intensely exploited and they yielded a dramatic development of mathematics in general and the calculus in particular. From reading (Kitcher, 1983) one cannot escape the feeling that the meanings associated to the idea of an infinitesimal were quite adequate to capture the underlying idea of “change” and to transform it into the body of knowledge we know today as calculus. In fact, there are good reasons to believe that the Leibnizian “signs” for the calculus and the formulation of change in terms of infinitesimals were responsible for the faster development of European continental mathematics when compared to its counterpart in England; see (Grabiner, 1997). In our view these arguments strongly suggest a clear cognitive advantage in “thinking change” in terms of infinitesimals and, also, explain why the chain rule in the language of infinitesimals is obvious to the point of not requiring explicit justification.

Kitcher (1983, p. 154) argues the contention that mathematical knowledge is cumulative when compared with scientific knowledge, which is regarded as being transformed in a more disruptive fashion. For instance, in resolving that the Lorenz transformations (which render invariant Maxwell’s laws of electromagnetism) are the basic transformations of physics, strictly

speaking, it is necessary to admit the incorrectness of Newton's laws, which remain invariant under the Galilean transformations. This is of course consistent with (Kuhn, 1970). In mathematics, on the other hand, in dealing with the famous error of (Cauchy, 1882) regarding the continuity of the limit of a series of continuous functions, analysis suffered a very profound transformation which brought about the  $\varepsilon$  -  $\delta$  definition of limits and the notion we know today as uniform converge. But, as opposed to physics, in mathematics, the previous body of knowledge of the calculus can be reformulated in terms of limits, and all of the "theorems" of the initial calculus continue to be valid in the new version of mathematical analysis that ensued from Cauchy, Weierstrass and others. Hence, in this sense, mathematical knowledge is cumulative. However, any teacher of calculus can attest to the fact of the great amount of difficulty that the Cauchy-Weierstrass theory of limits presents to students. This is perhaps to be expected as it took roughly a century from the time the initial calculus was invented to the formulation of the theory of limits to deal with Cauchy's "error". It thus seems reasonable to suggest that when paradigms change in mathematics (as the change towards the theory of limits after the infinitesimal approach) they must have a cognitive advantage for dealing with pressing unsolved problems, but this advantage does not necessarily extend for mathematics education. In fact, teaching the old body of mathematical knowledge with the new paradigms, in our view, adds a heavy overhead to the pedagogy of the subject matter.

Reflections related to observed advantages in student understanding when the calculus is presented in the language of infinitesimals appear in education journals; studies on this very topic, using Keisler's book *Foundations of Infinitesimal Calculus* (Keisler, 1976) as a textbook, have been made, and the observed results appear discussed in the literature (see, for example, Sullivan (1976)).

In (Kitcher, 1983, p. 155), the following qualified remark is found: “Unfortunately, the history of mathematics is underdeveloped, even by comparison with the history of science”. Clearly the topics for the explorations suggested by this brief exposition need a framework for the history of mathematics that lies beyond the bounds of semiotics, and these explorations are crucial for gaining a better understanding of the cognitive workings of the human mind as it strives to understand mathematics.

It would be, indeed, a framework that must include the discussion of pairings of signs and meanings validated by the history of mathematics as being effective, but it must also include the discussion of issues like the ones raised here. This framework, a sort of historical heuristics for mathematics education, should set the stage for exploring the cognitive workings of the human mind as it grapples with signs and meanings in its quest for advancing mathematical knowledge.

## References

- Campistrous, L. A., Lopez, J. M., and Rizo, C. (2009). *Reflexiones sobre la didáctica del cálculo a propósito de una lectura del primer texto publicado sobre esta materia por el Marqués Guillaume François Antoine L'Hospital: Analyse des infiniment petits pour l'intelligence des lignes courbes*, Submitted for publication.
- Cauchy, A. L. (1882). *Cours d'Analyse*, Série II, Tome III, Œuvres Complètes, 25 vols, Paris, 1882-1932.
- Cauchy, A. L. (1899). *Résumé des Leçons Données a L' Ecole Royale Polytechnique sur Le Calcul Infinitesimal*, Série II, Tome IV, Œuvres Complètes.
- Child, J. M. (translator), Leibniz, G.W. (Author). (2007). *The Early Mathematical Manuscripts of Leibniz*. Merchant Books.
- Dieudonné, J. (1960). *Foundations of Modern Analysis*. New York: Academic Press,
- Euler, L. (1748). *Introductio in analysin infinitorum*, Vol 1, Edición Facsimilar, SAEM “Thales”. Sevilla: Real Sociedad Matemática Española.
- Euler, L. (1748). *Introductio in analysin infinitorum*, Vol 2, Introduction to the Analysis of the Infinite (translation), Book II. New York: Springer Verlag.

- Euler, L. (1755). *Institutiones calculi differentialis*, Foundations of Differential Calculus (translation), New York: Springer Verlag.
- Fried, M. N. (2008). History of mathematics in mathematics education: A Saussurean perspective. *The Montana Mathematics Enthusiast*, 5(2&3), 185-198.
- Grabiner, J. (1997). Was Newton's calculus a dead end? The continental influence of Maclaurin's Treatise of Fluxions. *American Mathematical Monthly*, 104(5), 393-410.
- Keisler, H. J. (1976 ). *Foundations of Infinitesimal Calculus*. Creative Commons Attribution-Noncommercial-Share Alike, 2.5 Licence, <http://creativecommons.org/licenses/by-nc-sa/2.5/>.
- Kitcher, P. S. (1983). *The nature of mathematical knowledge*. London: Oxford University Press.
- Kuhn, T. S. (1970). *The structure of scientific revolutions*. Chicago: University Press.
- Lagrange, J. L. (1797). *Théorie des fonctions analytiques*. Paris : L'Imprimerie de la république.
- L'Hospital, G. F. A. (1696). *Analyse des infiniment petits, Pour l'intelligence des lignes courbes*. Paris: L'Imprimerie Royale.
- Polya, G. (1945). *How to Solve It: A New Aspect of Mathematical Method*. Princeton: Princeton University Press.
- Saussure, F. (1974). *Cours de Linguistique Générale*. C. Bally and Albert Sechehaye (eds.) with the collaboration of A. Riedlinger, Paris: Payot.
- Sullivan, K. (1976). The teaching of elementary calculus using the nonstandard analysis approach, *The American Mathematical Monthly*, 83(5), 370-375.
- Toeplitz, O. (1963). *The calculus: A genetic approach*. Chicago: The Mathematical Association of America and The University of Chicago Press.
- Wilkins, D. R. (Ed.). (2002). *The analyst; or a discourse addressed to an infidel mathematician*, Printed by and for S. Fuller at the Globe in Meath-street, and J. Leathly Book-seller in Dames-street. 1734,  
<http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.pdf>.



**-COMMENTARY-**

**Some reflections on Hernández and López's reflections on the chain rule**

Michael N. Fried<sup>1</sup>

Ben Gurion University of the Negev, Israel

To shed light on how history of mathematics can play a role in mathematics education, Hernández and López chose well in focusing on the chain rule as a case study. Generally speaking, it is, as they point out, a topic students find difficult to grasp: it is a rule involving a different order of complexity than, say, the rule for differentiating functions of the form  $f(x)=x^n$ , and, although it can eventually become intuitive, it is far from intuitive at the start. Harel *et al.* (2009) have pointed out that students' difficulties with the chain rule involve: "...coordinating three or more changing quantities [when viewed in a physical context]; executing the chain rule algebraically requires sophisticated abilities in recognizing algebraic form; making the translation between the physical and algebraic manifestations..." Others, reasonably enough, have seen the difficulties of the chain rule in conjunction with more general problems concerning the composition of functions (e.g. Cottrill, 1999). For their part, Hernández and López have looked for insight on the chain rule and how to confront the difficulties associated with it in one of the earliest instances where the rule seems to have been used, namely, in l'Hôpital's *Analyse des infiniment petits* (1696/1716).


L'Hôpital's point of departure was the notion that a curve could be thought of as an infinite number of infinitely small straight line segments, a polygon with an infinite number of sides. From that, it was an easy step for l'Hôpital to take the fundamental operation in the analysis of curves to be what he called the "difference," "the infinitely small portion by which a variable quantity increases or diminishes continually" (l'Hôpital, 1696/1716, p.2). For Hernández and López, accordingly, l'Hôpital's approach gives a historical justification for "...the cognitive advantages of defining the derivative as a difference arising from an infinitesimal change..." (p.6), that is, the cognitive advantages of a non-standard analysis framework, such as the one they used with success at the University of Puerto Rico. The connection is not absurd. On the one hand, Abraham Robinson, the founder of modern non-standard analysis himself recognized Leibniz as spiritual father of the subject, indeed, that his ideas vindicated Leibniz's (see Robinson, 1966, p.2). Robinson also mentions l'Hôpital's 1696 work, and l'Hôpital, himself, admits his own debt to Leibniz. So there is a certain undeniable confluence of

---

<sup>1</sup> [mfried@bgu.ac.il](mailto:mfried@bgu.ac.il)



thought here. At the same time, one needs to take great care in searching the past for ways of thought offering us “cognitive advantages.”

The danger is the temptation to make historical mathematical texts and historical mathematical thought work for us in ways that obscure their own idiosyncratic historical character. As I have pointed out elsewhere (Fried, 2008) and Hernández and López describe in their paper, semiotics can bring into relief the way the latter can occur, namely, that the synchronic structure of our sign system, in this case, how we employ and combine signs to speak and think about mathematics, presses and forces past mathematical thought, past synchronies into a modern mold. This also shows why the temptation—and, therefore, the danger—is so great and, sometimes, almost irresistible. It is so difficult to view the past on its own terms and not to see it as serving the present, a past that Michael Oakeshott has referred to as the “practical past” (see Oakeshott, 1999). Yet it is struggling against this temptation that historians of mathematics must forever engage in. Semiotics helps in this regard by making one aware of how meaning is constituted in the presence or absence of a sign or in how a sign is used with respect to other signs—and one should keep in mind that “function,” “rule,” “infinitely small,” are as much signs as  $f(x)$ , ,  $p$ , and  $\varepsilon$ ; in fact, even continuous texts can function as signs (Tobin, 1989). Hernández and López speak in their paper, in a fine turn of phrase, about finding a “historical heuristic for mathematics education” (pp.5,8). There is much to think about in formulating such a historical heuristic, but what we learn from semiotics is that this heuristic must at very least make it a guiding principle that how signs are used now is different than how they were used in the past and that the sign systems of the past were as integral and coherent then as ours are now. This is perfectly consistent with how historians see their own pursuit. As the historian Herbert Butterfield puts it: “...it is better to assume unlikeness at first and let any likenesses that subsequently appear take their proper proportions in their proper context; just as in understanding an American it is wrong to assume first that he is like an Englishman and then quarrel with him for his unlikenesses, but much better to start with him as a foreigner and so see his very similarities with ourselves in a different light” (Butterfield, 1951, p.38).

Hernández and López have certainly heeded this principle in highlighting l'Hôpital's reliance on infinitely small differences. They also pay attention to the crucial fact that the chain rule is never stated explicitly by l'Hôpital. The other crucial fact about l'Hôpital's *Analyse des infiniment petits*, which I think Hernández and López are aware of but do not mention, is that while l'Hôpital uses the word “variable” everywhere, he uses the word “function” nowhere. The two facts are, I believe, deeply related and give a clue as to the great differences between l'Hôpital's conception of the calculus and ours. These differences, in my view, ought to be stressed somewhat more than the apparent resemblance between l'Hôpital's infinitely small differences and Robinson's non-standard analysis, which, despite Robinson's own claims alluded to above remains moot. Let me, then, say a few words about these differences.

The complete title of l'Hôpital's work is *Analyse des infiniment petits pour l'intelligence des lignes courbes* (*Analysis of the infinitely small for the understanding of curved lines*). Although in the published work itself, the title becomes abbreviated to *Analyse des infiniment petit* and is subtitled *Du calcul des differences* (*On the calculus of differences*), it is the second part of the main title that reveals l'Hôpital's ultimate intention, namely,

the understanding of curved lines. And I use the word “intention” here almost in the Husserlian sense: curved lines provide the direction of l’Hôpital’s thought, and the understanding of curved lines and lines related to them informs the meanings of all his other terms. Thus the definition of a “variable” begins the book since a curve is given by an equation that relates, algebraically, the lengths of certain lines. Such lengths, or other cogent geometrical objects related to curves, are the variables l’Hôpital really has in mind even though in principle an algebraic equation can be a purely symbolic expression. If there is any doubt about this, just consider that immediately after defining “variables” and “constants” he writes, “Thus in a parabola the ordinates (*les appliquées [par ordre]*) and the lines cut off [from the diameter by the ordinate] are the variable quantities and instead of the parameter [the *latus rectum* of the parabola] is the constant quantity” (p.1). Immediately afterwards, moreover, when l’Hôpital defines the “difference,” the examples given are again related to a curve and are not only lengths, but also triangles as differences of segments of the curve and rectangles as differences of areas contained by the curve, diameter, and ordinates.

Curves and their equations, then, are always present in l’Hôpital’s view of his subject, sometimes implicitly, usually explicitly; his notion of infinitely small differences is suggested by his understanding of curves and, in turn, is meant to be a tool in furthering his understanding of curves. Functions, as I have said, are nowhere in l’Hôpital’s work. In Leibniz, whom l’Hôpital read and admired, the word “function” does appear, but, there too, it is only in relation to curves: functions are relations connected to tangents, normals, subtangents, subnormals, and so on. Adding the sign “function” in mathematics, which occurred in the following century, meant that the kind of calculus described by l’Hôpital could be thought of as being about the relations connected with a single abstract object, the function, rather than the almost endless variety of individual curves. In fact, if one sees the idea of an infinitesimal difference coming out of a conception of curves, one can understand why an approach to the calculus by way of functions is set against one inspired by infinitesimals. This can be seen strikingly in the title of Lagrange’s treatise on the calculus written exactly one century after l’Hôpital’s and partly a response to it (see the introduction to Lagrange, 1797): *Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toute considération d’infiniment petits ou d’évanouissans, de limites ou de fluxions, et réduits à l’analyse algébrique des quantités finies* (*Theory of Analytic Functions: Containing the Principles of the Differential Calculus Free of All Consideration of the Infinitely Small or Evanescent, Limits, or Fluxions, and Reduced to the Algebraic Analysis of Finite Quantities*) (1797).

Lagrange’s work presents the chain rule as a rule (Lagrange, 1797, p. 30), as Hernández and López point out. Lagrange can do this precisely because he has shifted the object from curve to function. What this shift allows, in effect, is an entire range of second-order concepts such as a “function of a function” and a “derivative of a function” (and it is worth reminding that in this same work, Lagrange invents the term “derivative” to mean the function *derived* from the “primitive function” from the series  $f(x+i)=f(x)+Pi+Qi^2+\dots$ ). One cannot express the chain rule as a rule without such second order ideas. It is a subtle point. l’Hôpital can speak of the kinds of first order relations among variables that produce equations of curves, relations such as  $x^2$ ,  $x^3$ ,  $xy$ , and for *these* he can provide rules; however, he cannot give rules for relations of relations, for it is curves and not relations themselves that form the domain of his intentionality. So,

it is only in examples of the rule for "perfect and imperfect" powers, that is, for integral and fractional powers, that we discern our chain rule in l'Hôpital's work. This rule, Rule IV, is the rule stating that the difference of  $x^m$  is  $mx^{m-1}dx$ , and the examples are as follows: the difference of  $(ay - xx)^3$  (the one quoted by Hernández and López), the

difference  $\sqrt{xy + yy}$ , the difference of  $\sqrt{ax + xx + \sqrt{a^4 + axyy}}$ , and finally,

$$\frac{\sqrt{ax + xx} + \sqrt{a^4 + axyy}}{\sqrt{xy + yy}}$$

One might argue that the chain rule is truly implicit in Rule IV since one can consider the difference of  $u^m$  which is  $mu^{m-1}du$  and then consider the difference  $du$  in terms of the difference  $dx$ . Of course we desperately want to say, "where  $u$  is a function of  $x$ ," but this, precisely, is what we cannot say.

The difficulty of these examples in comparison to the simple powers referred to in Rule IV itself give them more the air of problems based on the rule rather than simple examples of it. And in his solutions (even though these might be in fact more to Bernoulli's credit than l'Hôpital's) one senses that l'Hôpital is showing off—a kind of one-upmanship not at all untypical in 17<sup>th</sup> century mathematics. But this too brings us further away from the chain rule as a rule. If we do something that helps us solve a problem, a clever manipulation or representation, we generally do not see ourselves as following a rule. In fact, to the extent we say we are following a rule, we lessen our claim to be solving a problem. For a problem is a problem precisely when there are no means known in advance of solving the problem. We prove our problem-solving mettle by pulling something out of a hat.

To return to mathematics education, what is it really that we learn from l'Hôpital and the chain rule? It is not, I think, a different approach to the chain rule, even with his concentration on the infinitesimals with its seeming identity with the modern alternative of non-standard analysis. It is rather a different intentionality altogether with regards to basic objects of mathematical analysis. By studying l'Hôpital with an eye to the ways he differs from us, students and teachers can begin to see how the introduction of "function" can give rise to a "rule" as a rule. They begin to see how much our own way of talking about analysis—and therefore our understanding of analysis—is shaped by these differences. This kind of insight into our own understanding of things derived from looking at an understanding different than ours is in general what one wants learn from history and, no doubt, its clearest educational value. Let me then end by quoting Butterfield again writing about what we gain from history, or rather, what we lose when our history fails to distinguish past from present:

If we turn our present into an absolute to which all other generations are merely relative, we are in any case losing the truer vision of ourselves which history is able to give; we fail to realise those things in which we too are merely relative, and we lose a chance of discovering where, in the stream of the centuries, we ourselves, and our ideas and prejudices, stand. In other words we fail to see how we ourselves are, in our turn, not quite autonomous or unconditioned, but a part of the great historical process; not pioneers merely, but also passengers in the movement of things (Butterfield, 1951, p.63)

## References

- Butterfield, H. (1951). *The Whig Interpretation of History*. New York: Charles Scribner's Sons.
- Cottrill, J. F. (1999). *Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions*. Ph.D. Dissertation, Purdue University.
- Fried, Michael N., (2008). History of Mathematics in Mathematics Education: A Saussurean Perspective. *The Montana Mathematics Enthusiast*, 5(2), 185-198.
- Harel, G., Holdener, J., McCallum, W., McClure, J., Thompson, P. (2009). The Chain Rule. Presentation at the Institute for Mathematics and Education, *Workshop on Mapping the Calculus Curriculum*. Available at the website: [http://math.arizona.edu/~ime/2008-09/Calculus/Group6\\_0419.pdf](http://math.arizona.edu/~ime/2008-09/Calculus/Group6_0419.pdf) (accessed, 15 April, 2010).
- Lagrange, J. L. (1797). *Théorie des fonctions analytiques*. Paris: De l'Imprimerie de la République.
- L'Hospital, G. (1696/1716). *Analyse des infiniment petits pour l'intelligence des lignes courbes*, 2<sup>nd</sup> edition. Paris: François Montalant.
- Oakeshott, M. (1999). *On History and Other Essays*. Indianapolis, Indiana: Liberty Fund, Inc.
- Robinson, A. (1966). *Non-Standard Analysis*. Princeton: Princeton University Press.
- Tobin Y. (Ed), (1989). *From Sign to Text: A Semiotic View of Communication*. Amsterdam: John Benjamins Publishing Company



## The history of mathematics as a pedagogical tool: Teaching the integral of the secant via Mercator's projection

Nick Haverhals<sup>1</sup> & Matt Roscoe<sup>2</sup>  
Dept of Mathematical Sciences  
The University of Montana

*Abstract: This article explores the use of the history of mathematics as a pedagogical tool for the teaching and learning of mathematics. In particular, we draw on the mathematically pedigreed but misunderstood development<sup>1</sup> of the Mercator projection and its connection to the integral of the secant function. We discuss the merits and the possible pitfalls of this approach based on a teaching module with undergraduate students. The appendices contain activities that can be implemented as an enrichment activity in a Calculus course.*

**Keywords:** conformal mapping; history of mathematics; integrals; Mercator projection; rhumb lines; secant function; undergraduate mathematics education

### Introduction

There is no shortage of research advocating the use of history in mathematics classrooms (Jankvist, 2009). Wilson & Chauvot (2000) lay out four main benefits of using the history of mathematics in the classroom. Its inclusion “sharpens problem-solving skills, lays a foundation for better understanding, helps students make mathematical connections, and highlights the interaction between mathematics and society” (Wilson & Chauvot, 2000, p. 642). Bidwell (1993) also recognizes the ability of history to humanize mathematics. His article opens with description of mathematics instruction treated as an island students perceive as “closed, dead, emotionless and all discovered” (p. 461). By including the history of mathematics, “we can rescue students from the island of mathematics and relocate them on the mainland of life that contains mathematics that is open, alive, full of emotion, and always interesting” (p. 461). Marshall & Rich (2000) argue that the history of mathematics can be a facilitator for the reform called for by the NCTM.

In addition to the benefits mentioned above, Jankvist (2009) identifies more gains that can be had by using the history of mathematics. Among them are increased motivation (that can be found in generating interest and excitement) and decreased intimidation - through the realization that the mathematics is a human creation and that its creators struggled as they do. Jankvist (2009) also mentions history as a pedagogical tool that can give new perspectives and insights into material and even can serve as a guide to the difficulties students may encounter as they learn a particular mathematical topic. Marshall & Rich (2000) conclude their article by saying:

---

<sup>1</sup> nicolas.haverhals@umconnect.umt.edu

<sup>2</sup> roscoem@mso.umt.edu

To sum up, history has a vital role to play in today's mathematics classrooms. It allows students and teachers to think and talk about mathematics in meaningful ways. It demythologizes mathematics by showing that it is the creation of human beings. History enriches the mathematics curriculum. It deepens and broadens the knowledge that students construct in mathematics class. (p. 706)

This is by no means a comprehensive summary of the research advocating the use of history in mathematics. All of the research cited above, particularly Jankvist (2009), provides many more sources. Bidwell (1993) also mentions three ways of using history in the classroom. The first is an *anecdotal display*, which features the display of pictures of famous mathematicians or historical facts in the classroom. The second is to *inject anecdotal material as the course is presented*. Here, Bidwell is referring to making historical references to coursework while it is being covered. Barry (2000), however, cautions against letting the use of history limited to the use of anecdotes. The third use mentioned is to *make accurate developments of topics a part of the course*. This third use best describes the remaining contents of this article.

#### *Background of this research*

The current research evolved out of an assignment given to two of the authors (N. Haverhals & M. Roscoe) in a graduate level history of mathematics class at the University of Montana<sup>ii</sup> which mutated into the study that is currently being reported. The remainder of the article is a reporting of this research.

#### **Methodology**

The study sought to investigate the merits of employing a historical approach through the teaching and learning of the topic of the integral of the secant, a topic that is common to most second semester calculus courses at both the high school and university level. The integral of the secant played a key role in the development of the Mercator map in the 16<sup>th</sup> and 17<sup>th</sup> centuries. The map was a critical tool during the age of discovery due to the fact that it was a conformal projection of the globe onto the plane, that is, it projected the globe in such a manner as to preserve angles (at the cost of distorting lengths and areas). This property allowed mariners to navigate across large expanses of featureless ocean by following compass bearings that the map provided.

In preparation for the investigation, the authors conducted a review of pertinent literature. In particular, we sought material treating the subject of the Mercator projection that was easily translated into an educational setting where the integral of the secant is taught *through* the historical reenactment of its discovery. Furthermore, we wanted to construct a unit that could be realistically included in a traditional calculus course. Since these courses typically allow for little divergence from firmly established traditional content, we decided that the unit had to be brief, able to be employed in a single class meeting.

After reading a number of articles and several educational units which dealt with the role of the integral of the secant in the Mercator projection we set out to design and create our own activity. It was decided that two documents would be produced. The first document consisted of a "take

home” primer on the Mercator projection (which can be found in Appendix 1). This document “set the stage” for the investigation. In it we gave a brief description of the problem of conformal projection and motivated the historical need for such a map during the age of discovery. We included new terms such as “rhumb line”, “loxodrome” and “conformal” as well as introduced the key historical figure in the development of the map, namely, Gerhardus Mercator. We also included an example of the important role of the conformal map by demonstrating how a seaman’s bearing changes for a non-conformal plane projected map leading to errors in navigation.

The second document that we produced was conceived as the “in-class” exploration of the integral of the secant (Appendix 2). In this document, we hoped to lead students through a “historical reenactment” of the discovery of the integral of the secant motivated by a desire for mathematical description of the Mercator projection. The document first asked students to reason about the horizontal scaling of latitudes and then went on to describe the “mechanical integration” that was carried out by Edward Wright which determined the vertical conformal scaling. Students were asked carryout and compare the accuracy of two such approximating integrations. A proof of the closed form of the integral was provided with several missing steps and students were asked to complete the traditional proof. Finally, a number of extensions to the in class exploration asked students to investigate the way that distance is distorted by the projection.

A sample of 16 undergraduate students consisting of 9 males and 7 females participated in the study. The students were all mathematics majors who had completed their calculus sequence. Students were given the “take home” document one week before being asked to complete the “in-class” document. A period of two hours was scheduled for the in-class portion. Students completed the exercise in groups of two. The authors of the study circulated about the classroom, answering questions. At the end of the period, the completed documents were collected.

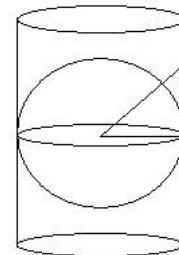
One week after participation in the classroom investigation into the historical account of the integral of the secant, two groups of four students each were chosen for separate case study analysis. One group of four students was interviewed to probe for affective reaction to the educational activity. These students were asked the following questions.

1. Describe what you learned in the activity on the historical approach to the integral of the secant.
2. How was the activity different from a typical mathematics class?
3. Here is the calculus textbook that we use here at the University of Montana. This is the presentation for the integral of the secant. How does it differ from the historical presentation of the integral of the secant that was presented last week?
4. Did the activity change the way that you view mathematical discovery?
5. Did the activity change the way that you view learning mathematics?
6. Would you say that you were more or less motivated to complete the traditional proof of the integral of the secant after having placed its discovery in a historical context?
7. Does including mathematics history make mathematics more meaningful? How?



Student response to these questions was audio recorded and transcribed for analysis. A second group of four students was shown a false physical model of the Mercator projection (chosen because of its commonality in supposedly “explaining” the projection). These students were asked to disprove the physical model using the knowledge that they had acquired through participation in the educational activity on the Mercator projection. Specifically, these students were shown the following:

A common misconception about the Mercator projection involves a physical model where the globe is projected onto a cylinder tangent to its radius through “illumination” of the globe from its center. Use the figure at the right to find the vertical stretching factor to determine whether or not this physical model gives rise to the Mercator projection.



Student response to this prompt was audio recorded and transcribed for analysis.

### Framework

While the use of history in mathematics classrooms is widely supported, it is probably safe to say that implementation is not so widely seen. Man-Keung Siu (2007) provides the following list of 16 *unfavorable factors* that contribute to the lack of history in mathematical classes:

- (1) “I have no time for it in class!”
- (2) “This is not mathematics!”
- (3) “How can you set question on it in a test?”
- (4) “It can’t improve the student’s grade!”
- (5) “Students don’t like it!”
- (6) “Students regard it as history and they hate history class!”
- (7) “Students regard it just as boring as the subject mathematics itself!”
- (8) “Students do not have enough general knowledge on culture to appreciate it!”
- (9) “Progress in mathematics is to make difficult problems routine, so why bother to look back?”
- (10) “There is a lack of resource material on it!”
- (11) “There is a lack of teacher training in it!”
- (12) “I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?”
- (13) “What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!”
- (14) “Does it really help to read original texts, which is a very difficult task?”
- (15) “Is it liable to breed cultural chauvinism and parochial nationalism?”
- (16) “Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?”

The list was compiled by Siu for the purpose of collecting the views of mathematics educators.

The authors took used their experience in preparing and administering their Mercator map activity to address each of these factors. The list was divided into groups of related items and these sub-lists form the next four sections.

### **A Philosophical Response to Three Unfavorable Factors**

Many of Siu's (2007) unfavorable factors for the use of history of mathematics in classroom teaching are tied to philosophical questions concerning the nature of mathematics and mathematics instruction. That is, in response to the query of, "Why don't you use the history of mathematics in your classroom?" teachers often disclose personal beliefs about mathematics and how it should be taught. Specifically the following list of three of Siu's unfavorable factors fit this description:

- (1) "I have no time for it in class!"
- (2) "This is not mathematics!"
- (9) "Progress in mathematics is to make difficult problems routine, so why bother to look back?"

By stating that there is no time for a historical approach to the teaching of mathematics in the classroom, teachers reveal personal beliefs that the history of mathematics is peripheral to other content matter in the subject which are given higher priority in classrooms where time is a limited commodity. This is especially the case in the modern American setting where student performance in mathematics on state and federally mandated tests is directly tied to school funding which places direct pressure on mathematics teachers to produce students who are computationally proficient in arithmetic, geometry, algebra and the like.

The statement "this is not mathematics" is a rejection of the history of mathematics as traditional mathematical classroom content. Here, the personal philosophy of mathematics might be seen as one which draws a clear line between that which is *history* and that which is *mathematics* thereby promoting a vision of mathematics that is at once highly specialized while also strictly compartmentalized from other areas of study.

Finally, the statement equating progress in mathematics with making "difficult problems routine" is a firm expression of a philosophy of mathematics which can be best described as one which seeks to avoid the complexities associated with the historical development of the subject in favor of routines, algorithms and memorized procedures.

While many authors have written about the role of personal philosophies in the teaching of mathematics (Thom, 1973; Hersh, 1986; Ball, 1988; etc), perhaps Paul Ernest's (1988) framework of philosophies of mathematics provide the most succinct and streamlined approach to the subject. Ernest identifies three psychological systems of beliefs about mathematics each with components addressing the nature of mathematics, the nature of mathematics learning and the nature of mathematics teaching.

Ernest identifies the *instrumentalist view*. Here the conception of mathematics is one of an “accumulation of facts, rules and skills to be used in the pursuance of some external end” (Ernest, 1988, p. 2). Mathematics is then thought of as a useful collection of unrelated rules and facts. The teacher’s role is then envisioned as an *instructor* who promotes skills mastery and correct performance in his or her students through strict adherence to curricular materials. The student of mathematics fulfills a role characterized by compliant behavior leading to the mastery of mathematical content, namely, the rules, skills and mathematical procedures presented by the teacher.

Ernest secondly describes the *Platonist view* of mathematics. Here the conception of mathematics is one of a “static but unified body of certain knowledge” (Ernest, 1988, p.2) which is discovered (not created) by humans through mathematical investigation. Thus mathematics is inherent to the world in which we live. It is a “universal language” which exists independently from human knowledge or awareness of the subject. The teacher’s role is then envisioned as an *explainer*, tasked with the promotion of conceptual understanding in his or her students as well as a presentation of mathematics as a unified system of knowledge. The student then learns mathematics through reception of mathematical knowledge. Proficiency is demonstrated through student presentation of knowledge possession, usually taking the form of variations of the same sorts of problems presented by the teacher during instruction.

Finally, Ernest describes the *problem solving view* of mathematics. Mathematics is conceived of as “a dynamic, continually expanding field of human creation and invention, a cultural product” (Ernest, 1988, p.2). Here mathematics is seen as a process rather than product, a means of inquiry rather than a static field of knowledge. As a human created body of knowledge, mathematics is envisioned as uncertain and open to refutation and revision. The teacher’s role is then taken as *facilitator* and is tasked with the confident presentation of problems. The student then learns mathematics through the act of problem solving, actively constructing knowledge through investigation. Proficiency in mathematics is equated with autonomous problem solving and even problem posing.

Placing the historical approach to the integral of the secant into this philosophical framework it seems apparent that our approach to this common calculus topic seems most strongly associated with the problem solving view of mathematics. The activity was presented to the student group with little more than an introduction concerning the problem of mapping a spherical globe onto a planar map. The questions posed were largely open-ended and lacked any algorithmic approach. The role of the teacher (here, the authors) was one of facilitator. Students were expected to construct their own knowledge through active investigation and group collaboration.

Perhaps more notable is the fact that the presentation of the discovery of the integral of the secant, as a necessary component of a conformal projection of the globe, can be thought of as a historical argument for the problem solving view of mathematics. Indeed, the first map presented by Mercator was produced without the aid of the integral – Mercator produced his map through geometric construction (Rickey, 1980). The map was improved upon by Wright through “mechanical integration” and the use of tables of values of the secant taken at one minute intervals (Sachs, 1987). Finally, the actual exact value for the integral of the secant was discovered by Henry Bond through the keen observation that Wright’s sums seemed to agree with tables of values of

$$\ln \left| \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \right|,$$

which can be shown to equal

$$\ln |\sec \theta + \tan \theta|,$$

the value of the integral of the secant that is presented in modern calculus textbooks today. Certainly this presentation of the subject presents a notion of mathematics that is “dynamic”, “continuously expanding” and “open to refutation and revision” as Ernest’s problem solving approach describes.

If we place each of the three of Siu’s unfavorable factors listed above into Ernest’s framework of mathematical philosophies it seems apparent that these objections are most closely aligned with the instrumentalist view of mathematics. “I have no time for it in class” seems to imply a classroom where the teacher’s role is taken as instructor (note the use of “I” instead of “we”). “This is not mathematics” seems to reject historical lessons on the basis that they do not promote any specific “skill”. Finally, the instrumentalist approach is especially apparent in the last comment that identifies “progress in mathematics” as making “difficult problems routine” which presents a truly procedural philosophy of mathematics and mathematics instruction.

While philosophical debate over the true nature mathematical knowledge continues, educators from both Platonist and problem-solving perspectives level criticism directed at the instrumentalist approaches to mathematics education. Indeed, Thompson (1992) notes that none of the philosophical models of mathematics education have “been the object of more criticism by mathematics educators than the model following most naturally from an instrumentalist perspective” (p.136). Critics of the approach argue that computational proficiency is not necessarily a measure of mathematical understanding and point to studies that document impoverished notions of mathematics by students who display satisfactory performance on routine tasks (Schoenfeld, 1985). Proponents of the problem solving view also object that the instrumentalist approach denies the student the opportunity to “construct” their own mathematical knowledge thereby disallowing the student true understanding of the structure in mathematics which is discovered through active investigation.

It seems evident that these three unfavorable factors for the use of a historical approach to the integral of the secant are actually subtle philosophical arguments concerning the nature of mathematics and mathematics instruction. When placed within Ernest’s framework of philosophies of mathematics it is apparent that the incorporation of such an activity into instruction on the topic most closely aligns with the problem-solving view of mathematics, while the reasons not to incorporate such an activity align most closely with the instrumentalist view of mathematics. Perhaps the debate is best concluded through deictic example by imagining a classroom where historical approaches to mathematics are strictly forbidden. In such a world,

the student would come away from a mathematics lesson with little notion of where mathematics comes from or how it is developed. There would be no sense of mathematics driven by both practical necessity and human curiosity, both of which play into the story surrounding the integral of the secant. Finally, there would be little appreciation for those that have given us the wealth of knowledge that we now enjoy or biographical inspiration to further the science.

If we have no time for history in mathematics instruction then we have abandoned crucial sources of inspiration and insight. If history of mathematics is not mathematics then mathematics is without a story: alien to the student, not of this world. And if mathematics is meant to “make difficult problems routine” then we should expect our students to excel only in that which is “routine” which certainly will not equip them with the tools to adapt in a changing world.

### **Student Responses to Unfavorable Factors**

Several of Siu’s (2007) unfavorable factors for the use of the history mathematics in classroom teaching relate to teacher’s beliefs regarding student’s opinions about the use of such materials in the classroom. The following four, in particular, fit this description:

- (5) “Students don’t like it!”
- (6) “Students regard it as history and they hate history class!”
- (7) “Students regard it just as boring as the subject mathematics itself!”
- (8) “Students do not have enough general knowledge on culture to appreciate it!”

Each of these reasons for not incorporating the history of mathematics into mathematics instruction proceeds from the standpoint of the student and argues against its incorporation into the mathematics classroom on two fronts.

The first three factors (5, 6, and 7) seem to argue that the inclusion of the history of mathematics in the mathematics classroom has a negative (or negligible) outcome on student motivation in the subject. Students who do not like or even “hate” the history of mathematics are likely to be unmotivated and even repelled by its inclusion in the classroom. If historical approaches to mathematical topics, such as the integral of secant, are “just as boring” as a more traditional approach then, it is argued, such approaches are perceived as a waste of a teacher’s valuable classroom and preparation time. These three factors seem to argue that the benefits of historical approaches to mathematical topics do not outweigh the costs that such approaches require of the teacher in terms of research, planning and implementation.

The last factor (8) is, perhaps, more severe than the first three. For here there is a tone of cultural superiority on the behalf of the teacher. The student is perceived as culturally deficient in their ability to perceive and understand mathematics when it is placed in a historical context. There is a tone of “teacher knows best” what is “good for the student” in terms of the lessons of history.

Our study, which placed the integral of the secant in a historical context by examining the development of the Mercator projection map, found evidence which dispels the factors provided by Siu which are outlined above. During the implementation of the unit, students displayed

intense curiosity in the mathematics behind the projection. With some instructional guidance, all student groups were able to successfully finish the unit in the two hour classroom time that was allotted for the experiment.

In a follow up to the activity, four students were chosen at random for interview which was conducted one week after the classroom meeting in which the experiment had been conducted. Each student was asked the following questions:

1. Describe what you learned in the activity on the historical approach to the integral of the secant.
2. How was the activity different from a typical mathematics class?
3. Here is the calculus textbook that we use in this mathematics department. This is the presentation for the integral of the secant. How does it differ from the historical presentation of the integral of the secant that was presented last week?
4. Did the activity change the way that you view mathematical discovery?
5. Did the activity change the way that you view learning mathematics?
6. Would you say that you were more or less motivated to complete the traditional proof of the integral of the secant after having placed its discovery in a historical context?
7. Does including mathematics history make mathematics more meaningful? How?

Transcripts from these interviews were analyzed for evidence for or against the merits of Siu's factors outlined above. All four interviewees were found to respond favorably to the historical approach to the integral of the secant. Consider the following response from student 1 to question 4:

I think that it's unbelievable, first of all. That people find these connections...and the fact that these guys did it without the tools that I have now. I mean, Mercator doing this, not perfectly, but pretty good, pretty good, having an idea, um, I just think that it's really cool that they know there's an answer. That these guys are so intelligent that they know something's up...and through their own intuition and through their own work they get there...and that has to be the greatest feeling ever for these guys. So, it gives more respect to anyone that has discovered something that we use or even something that we don't use in our Calc books...it definitely gave me more respect for these guys. It's unbelievable that they did these things...(Student 1)

Certainly the response to the question displays a sense of wonder at the use of mathematics in the Mercator projection. Words such as "cool" and "unbelievable" and "respect" are used in describing the historical discovery of the integral in the making of the map. In response to question 3, student 1 comments:

And, again, you can show me this and I am going to accept it, 'cause it's in my Calc book and we have no choice but to accept it and memorize it...but that is completely different than starting with this [points to historical approach to integral of secant activity]...starting with integration and ending with the natural log of the secant of  $x$  plus the tangent of  $x$ . So, for me and the way that I think and the way that I enjoy school, it

was helpful, and I can even imagine seeing myself start out learning about integration with this example (Student 1)

Again, student 1 responds favorably to the activity placing it in the category of activities that the student “enjoys” in school and calling it “helpful”. Student number 2 also expressed a positive reaction to the activity. In response to question 5 the student comments:

I found it that it made me feel that my work was more important than it usually is. And the fact that usually when you do a problem, you get an answer, and think you’re done, but, there really is no point to it that you see...I mean...when you’re taking...you’re doing integration by parts, it’s like, okay, what are we ever going to use this for? And so, you do all this work and you never see really ever where it applies...they’ll try to do stuff and...I mean it’s really, really basic and it doesn’t really apply, but, if they could take examples and show where its used, the historical context, it makes it feel as if you’re kind of working along side of those people when they were actually doing the work hundreds of years ago...you went through and saw what they did, and so it gives a level of importance that isn’t usually ever there...you know...that was valuable. (Student 1)

Here the student contrasts “traditional” approaches to common calculus topics (integration by parts) with the historical approach to the integral of the secant and describes how the historical approach lends a “level of importance” to an otherwise mundane mathematical topic. Student 3 in response to question 7 echoes this sentiment:

It’s nice to be able to first learn about the secant and then they’ll show you what it’s used for...then it makes a lot more sense because you have been exposed to it already and you’re already kind of familiar to it. It’s nicer to see people apply it to their life and situations. (Student 3)

Here we see the characterization of the historical approach as adding a real world “applied” aspect to instruction which is positively characterized as “nice” by the student. Finally, student 4, in response to questions 1 and 7, comments that:

I liked the historical context. It helps me put things in perspective. It’s cool that people were applying integration before integration was codified. It also illustrated the closer and closer estimations using smaller rectangles better than my Calculus study of Riemann sums... understanding how anything, especially math, is related to real world problems and solving them makes me more motivated to understand the methodology and consider broader applications of the problem solving technique. (Student 4)

And so, our analysis of student response to the historical approach to the integral of the secant is unanimous in its approval of the educational technique. Rather than Siu’s suggestion that they hate it, find it boring and non-motivating, our study group reported that they “enjoy” it and characterize the approach as “cool” and “useful” commenting that they are “more motivated” and “interested” in mathematics which is given an added “level of importance” when it is couched in a historical context. Furthermore, our data shows that students *do* have enough

cultural maturity to appreciate the approach. There is a sense that the accomplishment of a conformal map was an “unbelievable” achievement won through great intelligence with “the tools that they had” before the advent of calculus through mechanical integration. There is evidence of an understanding that these early map makers were “applying integration before integration was codified” which displays the student’s ability to imagine a mathematical culture before the invention of the calculus. Finally, there is a sense that the student is “working along side of those people when they were actually doing the work hundreds of years ago” which certainly indicates a level of cultural respect and admiration.

### **A Logistical Response to Unfavorable Factors**

As expected from a list as comprehensive as Siu’s, a number of the factors that discourage teachers from employing the history of mathematics deal with very practical matters. This list of unfavorable factors relating to practicality is divided into two groups: logistics and preparation.

The following list is comprised of the factors the authors would describe as logistical in nature. These are factors that might discourage even those who are inclined to include the history of mathematics in their teaching. Each factor in the list will be addressed individually, from the perspective of the authors and through the lens of creating and implementing the teaching module. The list of logistical factors, determined by the authors, is as follows:

- (3) “How can you set question on it in a test?”
- (4) “It can’t improve the student’s grade!”
- (13) “What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!”
- (14) “Does it really help to read original texts, which is a very difficult task?”

The first factor on the list, (3), is one that most teachers would probably agree needs to be addressed before using the history of mathematics in their classes. Rarely is anything presented by a teacher deemed unimportant. However, it is generally necessary to test students on material in order for them to see it as important. At the same time, asking students to be accountable for historical and/or biographical information is likely to re-enforce the sorts of ideas responsible for factor (2).

The trick, then, is to create assessment questions that use the skills developed in the course of using the history of mathematics. This can come in a number of forms. The first and most obvious is the form that assessment generally comes in. Often, when new material is presented in a class, the teacher will lecture for some period of time and then leave the students to practice the skills taught for the remainder of the class and on the assigned homework. However, this practice usually only matches a small portion of the lecture time – when the teacher presents examples, usually occurring at the end of the lecture. Typically the practice problems assigned does not match a bulk of the lecture – the part when the teacher explains, justifies or proves the technique or material to be taught. Thus if this portion of lecture time is spent presenting (explaining, justifying or proving) the material from a historical perspective, little need be changed in the way that students are assessed.



This response to factor (3) may easily be criticized, however, on the grounds that it still promotes the sort of “drill and kill” mentality that the inclusion of the history of mathematics is largely meant to discourage. If the goal for including the history of mathematics in the classroom is to get away from this mentality and promote the development of other skills (such as problem solving) then assessment should reflect this aim. For this, the preparers of the historical content need to get creative.

For our study, a problem very much related to the Mercator projection was chosen and given to four students during a follow-up interview a week after the study. The students were asked to evaluate the validity of a commonly used physical characterization of the Mercator projection. The illustration (see methodology section above) was described as follows: students were asked to imagine a semi-transparent globe sitting snugly in a cylinder with a light bulb glowing in the center of it. Light shines through the globe and “shadows” are cast by land masses on the globe. These shadows become the placement of the land masses on the cylinder, which is then sliced and laid flat to form the map. While this characterization does share some properties with the Mercator projection (the poles of the globe can never be projected and the stretching increases with increases in latitude), it is actually a different (non-conformal) projection. As the students were able to deduce, the factor of stretching in this alternative representation is a tangent function which is not equal to

$$\ln|\sec \theta + \tan \theta|,$$

the factor of stretching found in the exploration.

While the students definitely needed some nudging to get them started, once the ball was rolling all four were able to deduce that the representation would not yield the Mercator projection. The authors feel that a question such as this would make for a legitimate test question. Granted, the students did need encouragement and some may not have known where to start if they saw it on a test they had to work out on their own. However, the authors believe that this is due in large part to the fact that the students are rarely asked to perform this type of task. If mathematics was taught from more of a historical perspective, they would be more accustomed to problem-solving and therefore would be more flexible in their thinking. It is worth noting that the first part of the Mercator exploration asked the students to find the factor of stretching for arbitrarily chosen latitude. The potential assessment question had the students do the same thing from a different perspective – so it indeed was assessing a skill they used in the activity.

The second item on this sub-list of unfavorable factors, (4), deals with students’ grades. Like before, the way in which using the history of mathematics in the classroom affects students’ grades depends on how it is used. If it is used simply as an alternative lecturing format with little or no change in assessment then it is possible that students see no benefit in terms of grade. However, the history of mathematics can be used as a guide for how students learn mathematics. This can be seen in the difficulties students have in learning particular mathematical ideas (Moreno-Armella & Waldegg, 1991; Jankvist, 2009) and in students’ conceptions of mathematical proof (Bell, 1976; Almeida, 2003). By using history as a guide for how students learn, it is possible that instruction could be improved.

This can even be taken a step further. Fawcett (1938/1966) describes a high school geometry class which, it could be argued, was set up in a fashion that mimics the historical development of Euclidean geometry. Under the guidance and supervision of the teacher and

through class discussion and consensus, the experimental geometry class created their own textbooks consisting of definitions, axioms and theorems. The main goal of the experiment was to improve the students' knowledge of mathematical proof, a goal that was achieved. It should be noted, however, that the students also outperformed a control class on a standardized geometry test administered state-wide. This was despite the fact that the experimental class covered less material than the control class. Some of this uncovered material showed up on the standardized test, but the students in the experimental class were flexible enough to deal with material new to them.

The last two unfavorable factors in this section, (13) and (14), are quite similar and will be addressed together. Basically, they are both speaking to the fact that dealing with historical mathematics can be quite difficult. Much of the time, this is true. While modern day mathematicians can often handle the mathematical content associated with historical mathematical documents, other barriers to understanding exist. One stumbling block stems from the fact that the first solution to a problem is rarely the most elegant or straightforward to understand, as mentioned in factor (13). Language (terminology) and notation are two other major obstacles, referred to in (14).

The authors believe, however, that the module provided serves as an example that these concerns can be addressed. Although the authors made every effort to make the activity historically accurate, much of the historical difficulty was described, rather than recreated. Students were asked to mimic the process of mechanical integration (before they likely realized that was what they were doing) used historically but to a far less accurate, but more user friendly, degree. Based on student responses this served the intended purpose, as each group was able to recognize that smaller intervals gave better approximations. This was a necessary insight to understand the link between the Mercator projection and the integral of the secant. Also, students were told about the "lucky accident" that resulted in the discovery of the closed form for the integral of secant; they were not expected to find it on their own. Relieving the students of unnecessary difficulty does not mean they are left with nothing to do on their own. As is mentioned in the module, the original proof for the validity in question was extremely laborious and difficult. The students were then guided through an alternative (and later) historical proof – one that allowed the use of methods familiar to the students from their pre-calculus and calculus classes.

The amount of editing of historical material virtually eliminates the factor (14) from the students' perspective. The only original material that made it into the final teaching module was quotes carefully chosen to provide historical context (or humor, as the case may be). Factor (14) is not yet eliminated from the content-preparer's perspective. However, this will be addressed in the next section.

### **A Response to Unfavorable Class Preparation Factors**

Factor (14) raises a completely different issue from the teacher's point of view. If the students can be shielded from difficult to read original texts, are not the teachers responsible for doing the shielding? Not completely, as was seen by the authors in the preparation of the teaching module. This issue is tied into the next three factors that Siu mentions. They are:

(10) "There is a lack of resource material on it!"

(11) “There is a lack of teacher training in it!”

(12) “I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?”

The bulk of the material that made its way into the activity came from journal articles or other teaching modules relating to the topic. In this way, the authors were not responsible for dealing with the difficult task of reading original material. Rather, they were free to concentrate on preparing the material in such a way as to be appropriate for their students.

This speaks to factor (10) as well. In preparing the module, the authors found more than enough resource material on the topic. It is true that not all of the material was deemed suitable by the authors for their targeted students. However, the materials found did provide enough for a complete, coherent teaching module to be put together. It should be noted that the authors acknowledge the possibility that factor (10) has not been interpreted as Siu intended. It is possible that what is being referred to is a lack of ready-to-use materials that can be implemented by teachers with little or no modification. The authors can not speak to this concern directly. Although some of the materials used were indeed designed to be used without modification, as mentioned, none were deemed appropriate for the students who were to see it. This was of no concern to the authors, however, because the creation of the module was an end in and of itself. Appropriate, ready-to-use materials were not sought. Instead, enough material was collected to complete the activity and that is all. It is unclear whether or not a completed module that was appropriate for the students in question could have been found.

The experience the authors had while completing this activity also helps dispel factor (11). While the module was originally meant to be part of a history of mathematics course the authors were taking at the time, no skills were explicitly taught in the class that lent themselves to its creation. What was gained from the class, however, was an appreciation for and interest in the history of mathematics. This new motivation, coupled with the authors’ existing mathematical skills, was sufficient to see them through to the completion of the project. As the activity was designed for students taking Calculus II, the authors feel that it (or something similar) could have been created by any teacher with a grasp of Calculus II material and the desire and interest to do so. Thus, in general, a lack of training in the history of mathematics need not be a deterrent for those teachers who wish to use it in their classrooms.

The last factor to be addressed in this section, (12), that will be addressed in this section is related to (11). One may get the feeling that since he or she lacks training in the history of mathematics, they may be ill-equipped to judge the accuracy of sources. The authors were able to alleviate this concern through the use of articles from reputable scholarly journals. The use of such journals assures the readers (content-producers) that the materials have been peer-reviewed. That way, the burden of verification is placed on professionals and the teachers preparing the material can concentrate on making it appropriate for and useful to their students.

### **A Response to the Final Two Unfavorable Factors**

The authors thought that the last two factors did not relate closely with the others and will be addressed briefly here. They are:

(15) “Is it liable to breed cultural chauvinism and parochial nationalism?”

(16) “Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?”

The first of these, (15), speaks to the potential that the history of mathematics has to create a classroom setting that is not agreeable to the teacher. It is possible that the history of mathematics could be used to create a narrow view of the development of mathematics. This narrow view, in turn, may lead to the impression that a select group of peoples alone were responsible for (and therefore good at) mathematics. This can easily be avoided by the careful inclusion of mathematics from many different cultures. While the contributions of the ancient Greek and later European cultures are well known, they are not the only wells from which to draw. The articles referenced by Katz (1995) and Wang (2009) serve as examples of articles that describe methods developed by other cultures.

The last unfavorable factor, (16), is one to which the authors can not respond. To their knowledge, there is no convincing empirical evidence that students learn better when the history of mathematics is used in the classroom. However, the student responses gathered from this article suggest that students would welcome the inclusion of the history of mathematics – and more enthusiastic students generally make for better learners.

### **Conclusion**

Siu has done the mathematics educational community a service by playing the role of devil’s advocate in maintaining a list of popular reasons why teachers do not use historical approaches in mathematics education. His list provided the framework for analysis of this educational experiment on the historical approach to the integral of the secant in the development of the Mercator projection map.

We found that several of Siu’s unfavorable factors could be characterized as subtle philosophical statements regarding the nature of mathematics and mathematics instruction. When viewed within the framework of Ernest’s (1988) philosophies of mathematics it is apparent that these objections most closely align with an instrumentalist view of mathematical knowledge and mathematical instruction. This view equates computational proficiency with mathematical understanding and is subject of much criticism in denying true mathematical understanding. In contrast, the historical approach employed in this study placed problem solving at the heart of instruction. By taking a historical approach to the subject, students learn that the closed form of the integral of the secant was “needed” to mathematically explain the Mercator projection. A historical approach allows for crucial sources of inspiration, insight and motivation which are missing from strict instrumentalist approaches, seen in this light, any argument against historical approaches can be seen as an argument in favor of an impoverished notion of mathematics.

A number of Siu’s unfavorable factors were characterized as teacher statements regarding negative student predispositions to historical approaches in the mathematics classroom. Analysis of our data disproves these notions. Student response to the activity was universally positive thus affirming the approach from a student standpoint and dispelling misapplied characterizations commonly held by teachers.

There were unfavorable factors that were seen as logistical concerns. Our unit demonstrates that each of these concerns can be overcome. We were able to creatively “set a question on a test” to the historical approach. We feel that, in the area of problem solving, the historical approach does “help student’s grades” by endowing them with a richer and more meaningful understanding of the process of mathematical meaning making. Finally, student difficulty in confronting historical text can be alleviated by careful and thoughtful presentation that is at once historically accurate while educationally streamlined toward an intended goal, in this case, an understanding of the integral of the secant.

In terms of Siu’s unfavorable classroom preparation factors, our study has shown to dispel many of the commonly espoused concerns. We encountered ample resources that aided the creation of the educational unit. No special teacher training was required. Lastly we appealed to reputable journals to insure accuracy of historical exposition, thus, an educator need not be a professional historian of mathematics in order to create educational materials which teach mathematical concepts from a historical standpoint.

While we acknowledge the concerns of cultural chauvinism and parochial nationalism raised by Siu, we feel that an evenhanded approach to historical topics in mathematics education may lead to quite the opposite outcome. Here the “historically educated” student of mathematics might come to an awareness of the great cultural and national diversity that has contributed to the development of the subject.

Finally, in response to Siu’s assertion that there is a lack of empirical evidence that supports historical approaches to mathematics in terms of improving student understanding we stand by the fact that student response to the unit pointed to greater interest and enthusiasm in the subject, which, we assert, are prerequisites to deep and meaningful learning in mathematics.

## References

- Almeida, D. (2003). Engendering proof attitudes: Can the genesis of mathematical knowledge teach us anything? *International Journal of Mathematical Education in Science and Technology*, 34(4), 479 – 488.
- Ball, D. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics*, 8(1), 40-48.
- Barry, D.T. (2000). Mathematics in Search of History. *Mathematics Teacher*, 93(8), 647- 650.
- Bell, A. W. (1976). A study of pupils’ proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7(1/2), 23 – 40.
- Bidwell, J. K. (1993). Humanize Your Classroom with The History of Mathematics. *Mathematics Teacher*, 86, 461-464.
- Ernest, P. (1988). *The impact of beliefs on the teaching of mathematics*. Paper presented at ICME VI, Budapest, Hungary.
- Fawcett, H. P. (1966). *The nature of proof: A description and evaluation of certain procedures used in senior high school to develop an understanding of the nature of proof*. New York, NY: AMS Reprint Company.

- Hersh, R. (1986). Some proposals for revising the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 9-28). Boston: Birkhauser.
- Jankvist, U. T. (2009) A characterization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics*, 71(3), 235 – 261.
- Katz, V. J. (1995). Ideas of calculus in Islam and India. *Mathematics Magazine*, 68(3), 163-174.
- Marshall, G. L., & Rich, B. S. (2000). The Role of History in a Mathematics Class. *Mathematics Teacher*, 93 (8), 704-706.
- Moreno-Armella, L., & Waldegg, G. (1991). The Conceptual Evolution of Actual Mathematical Infinity. *Educational Studies in Mathematics*, 22, 211–231.
- Rickey, V. F. & Tuchinsky, P. M. (1980). An application of geography to mathematics: history of the integral of the secant. *Mathematics Magazine*, 50(3), 162-166.
- Sachs, J. M. (1987). A curious mixture of maps, dates, and names. *Mathematics Magazine*, 60(3), 151-158.
- Schoenfeld, A. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press, Inc.
- Siu, M.-K. (2007). No, I don't use history of mathematics in my class. Why? In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings HPM2004 & ESU4* (revised edition, pp. 268–277). Uppsala: Uppsala Universitet.
- Sriraman, B., Roscoe, M., & English, L. (2010). Politicizing Mathematics Education: Has Politics gone too far? Or not far enough? In B. Sriraman & L. English (Eds), *Theories of Mathematics Education: Seeking New Frontiers* (pp. 621-638), Springer, Berlin/Heidelberg.
- Thom, R. (1973). Modern mathematics: Does it exist?. In A. G. Howson (Ed.), *Developments in mathematical education: Proceedings on the Second International Congress on Mathematics Education* (pp. 194-209). Cambridge: Cambridge University Press.
- Thompson, A. G. (1992). Teachers' beliefs and conceptions: a synthesis of research. In D. A. Grouws Ed., *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.
- Wang, Y. (2009). Hands-on mathematics: Two cases from ancient Chinese mathematics. *Science & Education*, 18(5), 631 – 640.
- Wilson, P.S., & Chauvot, J. B. (2000). Who? How? What? A Strategy for Using History to Teach Mathematics. *Mathematics Teacher*, 93(8), 642-645.

## Appendix 1

### Student take home

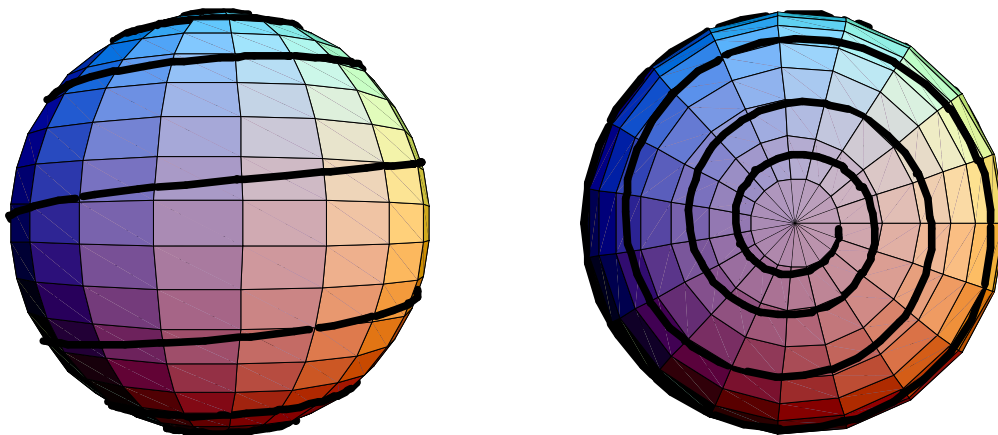
#### Mercator's World Map

#### A Historical Approach to the Integral of the Secant

Suppose that you are tasked with navigating a ship that is to travel from a point in Europe to the “New World” recently discovered across the Atlantic Ocean. How would you navigate the vessel using only 16<sup>th</sup> century technology? Most mariners during the age of discovery steered their ships along lines of constant bearing using a magnetic compass. A path of constant bearing on the globe is called a “rhumb” line named for the Spanish *rumbo* meaning “way” or “direction”. This concept of a path of constant bearing was later named a loxodrome from the Latin *loxos* signifying “slant” and *drome* signifying “running”. So, most mariners of the 16<sup>th</sup> century travelled paths across the ocean that we know call loxodromes.

On the globe a loxodrome intersects all north-south lines of constant longitude at the same angle. Parallels, or lines of constant latitude, are therefore loxodromes because they intersect all north-south lines of constant longitude (meridians) at right angles. Early sailors, cartographers and later mathematicians realized that these paths of constant bearing became spiral-like curves whenever the direction chosen was not due east or west. This effect is due to the fact that as a rhumb line moves north the distance separating meridians grows closer and thus the line must turn away from the pole to maintain the heading.

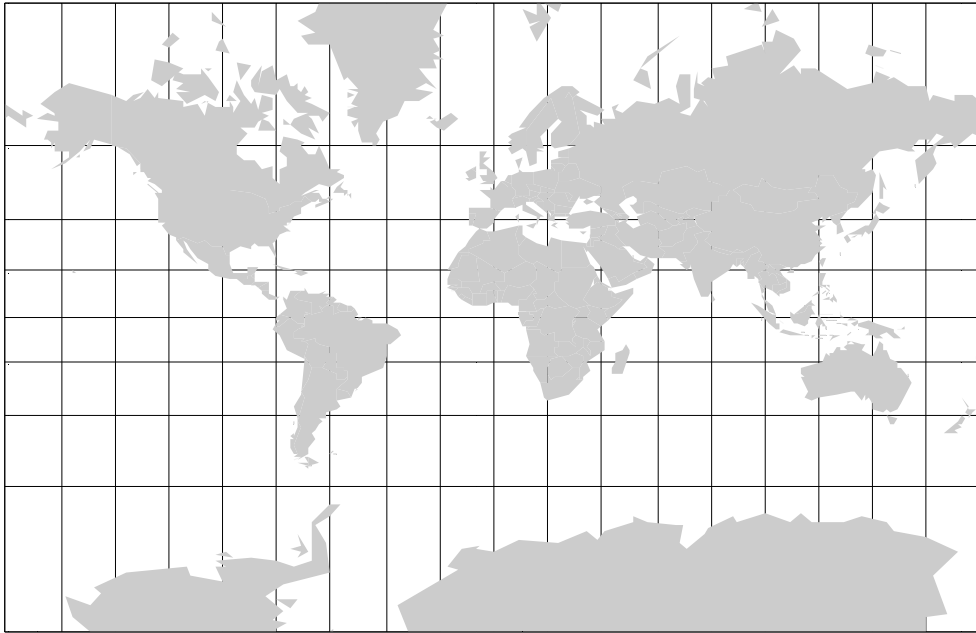
**Figure 1:** Two views of a typical rhumb line, a path of constant bearing, on a globe. All rhumb lines, except paths of constant latitude, create spiral paths on the globe differing only in slope.



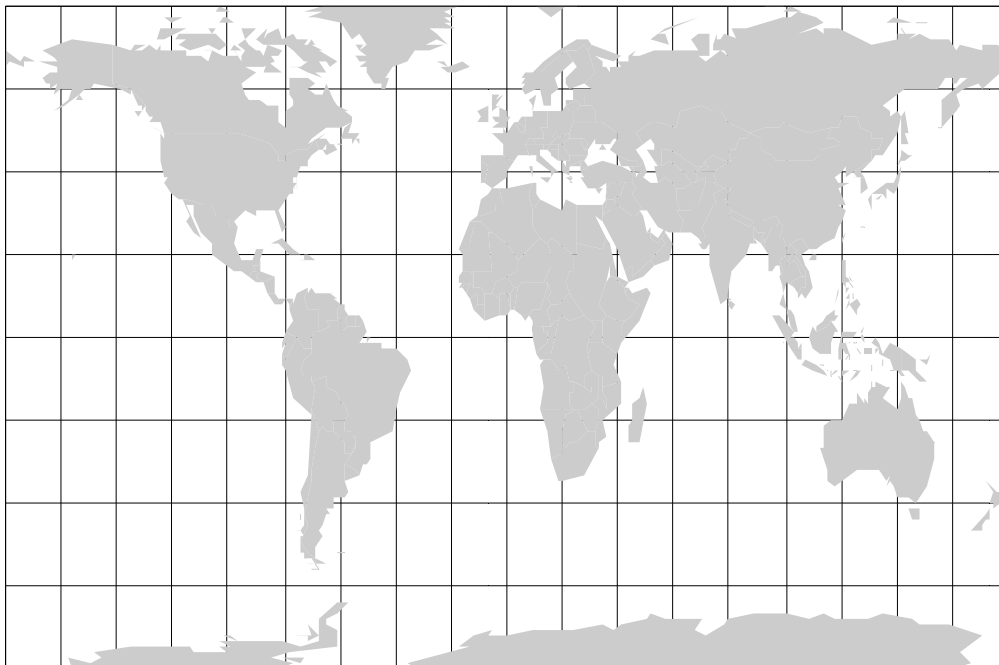
The spiraling nature of lines of constant bearing created the need for a special kind of map in which a sailor could draw a line from his present location to his objective and measure the bearing by determining the angle that is formed by the path and the meridians that are crossed in route. Such a map was presented to the world in 1569 by Gerhardus Mercator and is today

known as the Mercator projection. The map signified a gigantic improvement over previous plane projection maps and is still widely used in navigation today.

**Figure 2:** *A Mercator Projection Map*



**Figure 3:** *A Plane Projection Map*

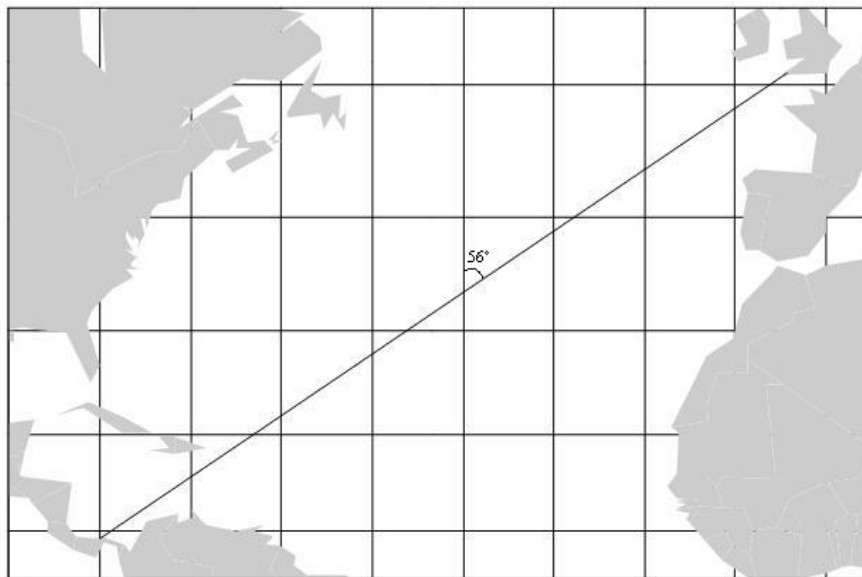




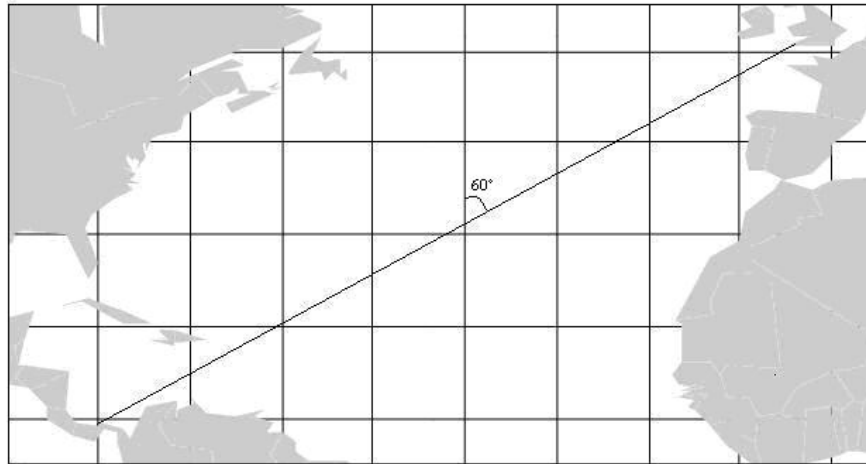
Close inspection of the two maps reveals that the plane projection has evenly spaced lines of constant latitude. In contrast, the distance between lines of constant latitude grows as a function of distance from the equator in Mercator's version.

In order to better understand the effect of Mercator's special scaling, consider a path on the globe that carries a seaman from Colon, Panama to Land's End, England. Using a magnetic compass (or the North Star) a sailor can successfully make such a trip by following a rhumb line that leaves Colon at a bearing of approximately  $56^\circ$  from true north. Such a path of travel over the globe then crosses all meridians at this same angle and thus scribes a spiraling loxodrome across the surface of the globe. If we were to plot the path of such a journey on both the Mercator and plane projection maps we would find that only on the Mercator projection would such a journey actually cross all meridians at an angle of  $56^\circ$  from true north thus correctly directing the sailor to his home (figure 4). On the plane projection map such a journey crosses all meridians at an angle of  $60^\circ$  from true north (figure 5). If we plot a course that leaves Colon at  $60^\circ$  from true north we find that our sailor is erroneously directed to France as indicated on the Mercator map (figure 6).

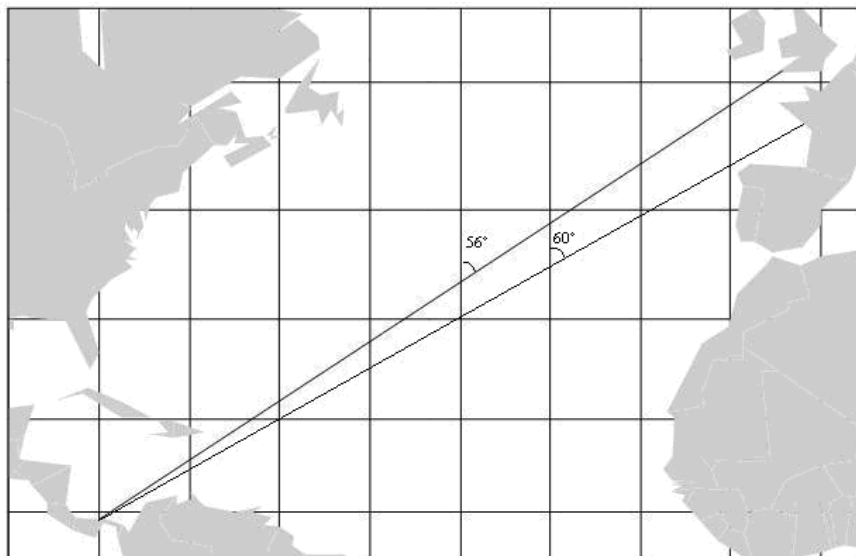
**Figure 4:** *Our Seaman's Journey on the Mercator Projection Map: Directs Seaman to a Bearing of 56 Degrees East of True North*



**Figure 5:** *Our Seaman's Journey on the Plane Projection Map: Directs Seaman to a Bearing of 60 Degrees East of True North*



**Figure 6:** *Our Seaman's Journey on the Mercator Projection Map: Bearing 60 Degrees and Bearing 56 Degrees East of True North*



So, it becomes apparent that the Mercator projection provides the seaman with a much more useful tool where a line of constant compass direction corresponds to a straight line on the map which making it possible for a 16<sup>th</sup> century seaman to determine the correct line of bearing to follow in order to arrive at the intended destination.

From a mathematical point of view, Mercator's projection is *conformal*, meaning that the projection from the globe onto the plane preserves angles. It should be apparent that the projection does not preserve distances. It is a mathematical fact that any projection from the sphere onto the plane cannot preserve both of these quantities, but that is another story best saved for another day. How did Mercator decide on his special scaling? Mercator himself comments on this scaling in the legend of the map of 1596:

In view of these things, I have given to the degree of latitude from the equator towards the poles, a gradual increase in the length proportionate to the increase of the parallels beyond the length which they have on the globe, relative to the equator. (Sachs, 1987)

Mercator created his special map using a compass and straight edge but mathematicians of the era challenged "any one or more persons that have a mind to engage" to mathematically describe the scaling that produced the successful map. (Rickey, 1980)

### References

- Alexander, J. (2004). Loxodromes: a rhumb way to go. *Mathematics Magazine*, 77(5), 349-356.
- Carslaw, H. S. (1924). The story of Mercator's map. *The Mathematical Gazette*, 12(168), 1-7.
- Comap. (N.D.). Calculus of the Mercator map. Derive Lab Manual for Calculus. Houghton Mifflin Co., New York, NY.
- Rickey, V. F. & Tuchinsky, P. M. (1980). An application of geography to mathematics: history of the integral of the secant. *Mathematics Magazine*, 50(3), 162-166
- Sachs, J. M. (1987). A curious mixture of maps, dates, and names. *Mathematics Magazine*, 60(3), 151-158.
- Tuchinsky, P. M. (1987). Mercator's world map and the calculus. In *UMAP modules in undergraduate mathematics and its applications*. Lexington, MA: COMAP.

## Appendix 2

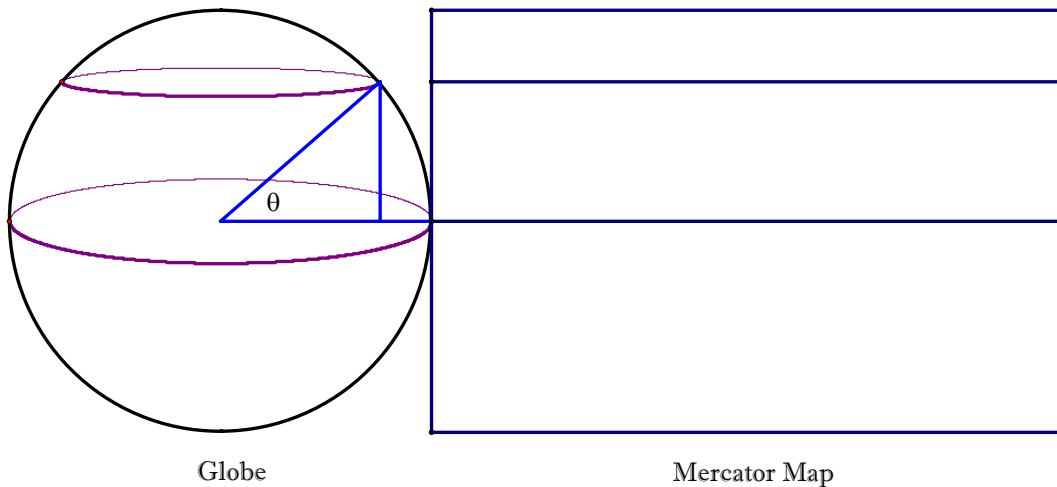
### Mercator in class activity

#### Mercator's World Map

#### A Historical Approach to the Integral of the Secant

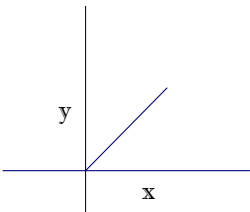
Mercator wrote, “In making this representation of the world we had...to spread on the plane the surface of the sphere in such a way that the positions of places shall correspond on all sides with each other both in so far as true direction and distance are concerned and as concerns correct longitudes and latitudes...With this intention we have had to employ a new proportion and a new arrangement of the meridians with reference to the parallels...It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator.” (Rickey, 1980)

Using the figure provided below determine the function that governs, “The lengthening of the parallels with reference to the equator.” That is, given a parallel at latitude  $\theta$  determine the function  $f(\theta)$  that tells us how the latitude lines must be stretched horizontally in order to appear equal in length to the equator.

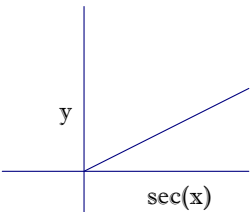


In the previous quote Mercator comments, “...It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator...” Mercator determined this vertical scaling through compass constructions. It was not until 1610 that Edward Wright, a Cambridge professor of mathematics and a navigational consultant to the East India Company, described a mathematical way to construct the Mercator map which produced a better approximation than the original. In 1599 he published *Errors in Navigation Detected and Corrected*. Wright argued that in order to preserve angles on the Mercator projection, the vertical scaling factor had to be the same as the horizontal scaling factor. To visualize this phenomena imagine a 45° angle drawn on a small portion of a globe. Recall that when this region gets projected to the plane, it gets stretched in the horizontal direction by an amount that depends on the latitude. Notice what happens. The angle as projected is no longer 45°. In order for the angle to be preserved, a stretch must occur in the vertical direction that matches the horizontal stretch.

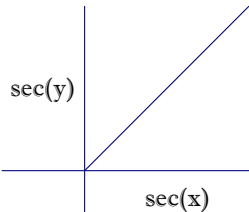
Angle on Globe



Horizontal Scaling



Conformal Scaling



Wright also realized that the correct interval of placement of a parallel on the Mercator projection was the result of the addition of any subintervals into which it could be divided. To this end, Wright made a table of secants taken at a common interval, added these results and then multiplied by the interval widths to determine the location of a particular parallel on the map. So, if the location of the 60<sup>th</sup> parallel is desired and an interval width of 10° is used then Wright would have performed the following:

Table of Secants	Multiply by Interval Width	Location on Mercator Map
Secant 10° = 1.0154	10° × 8.0954 = 80.954	Place the 60 <sup>th</sup> parallel at a location that is 80.954° north of the equator.
Secant 20° = 1.0642		
Secant 30° = 1.1547		
Secant 40° = 1.3054		
Secant 50° = 1.5557		
Secant 60° = <u>2.0000</u>		
Total = 8.0954		

Use Wright's method to determine the location of each of the following parallels using an interval length of 5 degrees.

Latitude on the Globe	Location on Mercator Projection
15°	
30°	
45°	
60°	
75°	
90°	

You should notice that the location of the 60<sup>th</sup> parallel that you just calculated is different than the one that was calculated in the example that proceeded. Which placement produces a more accurate map? How do you know?

What difficulty did you encounter in determining the placement of 90° Latitude? What is this location on the Globe? What are the implications for the Mercator Map?

Historically, Wright's table of secants had an interval width of one minute or one sixtieth of a degree. Describe mathematically, using modern notation, the process that Wright is carrying out in determining the vertical scaling of the Mercator projection. Is Wright's method exact? How could it be improved?

As you have probably discovered, the exact mathematical explanation for Wright's technique in developing the vertical scaling for the Mercator projection hinges on a closed form for the integral of the secant. In Wright's time this result was still some 50 years from being discovered. However, with Wright's charts at his disposal, Henry Bond in the 1640s had a very lucky accident. Bond, who fancied himself a teacher of navigation and mathematics, compared Wright's table to a table of values in which the tangent function was composed with the natural logarithm. This led him to conjecture that the closed form for the integral of the secant equaled

$$\ln \left| \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \right|,$$

which can be shown to equal

$$\ln |\sec \theta + \tan \theta|.$$

The first proof of the integral of the secant was provided in 1668 by James Gregory. Edmund Halley commented on the proof, “The excellent Mr. James Gregory in his *Exercitationes Geometricae*, published Anno 1668, which he did not, without a long train of consequences and complication of proportions, whereby the evidence of the demonstration is in a great measure lost, and the reader wearied before he attain it.” (Rickey, 1980). And so we avoid this proof and instead offer guidance through a proof offered by Isaac Barrow. Complete the missing steps in the proof

$$\begin{aligned}
 \int \sec \theta d\theta &= \\
 &= \\
 &= \int \frac{\cos \theta}{(1 - \sin \theta)(1 + \sin \theta)} d\theta \\
 &= \\
 &= \\
 &= \\
 &= \frac{1}{2} \int \frac{\cos \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 + \sin \theta} d\theta \\
 &= \\
 &= \\
 &= \frac{1}{2} [-\ln|1 - \sin \theta| + \ln|1 + \sin \theta|] + c \\
 &= \\
 &= \frac{1}{2} \ln \left| \frac{(1 + \sin \theta)^2}{(\cos \theta)^2} \right| + c \\
 &= \\
 &= \ln|\sec \theta + \tan \theta| + c
 \end{aligned}$$



The angle  $\theta$  in the previous integral assumes a radian measure. If  $\theta$  is measured in degrees then a change of variables will yield the following:

$$\int_0^\theta \sec \theta \, d\theta = \frac{180}{\pi} \ln |\sec \theta + \tan \theta|$$

Use this result to determine the exact location of each of the following parallels.

Latitude on the Globe	Location on Mercator Projection
15°	
30°	
45°	
60°	
75°	
90°	

How do these placements compare to those found earlier in the exercise?

## EXTENSIONS

1. How do distances vary on the Mercator map relative to latitude? Consider travelling parallel to the equator at various latitudes on the map. Consider travelling perpendicular to the equator at various latitudes on the map.
2. How do areas vary on the Mercator map relative to latitude? What happens to the area of a region as one moves farther from the equator? Give examples.

---

<sup>i</sup> The mathematics behind the Mercator map has nothing to do with the way the map ended up being used for political purposes. A number of critical theorists who have no idea of the mathematics behind the map run around saying “the map was purposefully made that way” . Gerhardus Mercator (1512-1594) created the map for navigational purposes with the goal of preserving conformality, i.e., angles of constant bearing crucial for plotting correct navigational courses on charts . In other words a line of constant bearing on a Mercator map is a rhumb line on the sphere. Conformality as achieved by Mercator with his projection came at the price of the distortion that occurred when projecting the sphere onto a flat piece of paper. The history of the map is also linked to the limitations of the Calculus available at that time period, and the difficulty of integrating the secant function (see Carslaw, 1924). Mercator himself comments, “...It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator...” Mercator determined this vertical scaling through compass constructions. It was not until 1610 that Edward Wright, a Cambridge professor of mathematics and a navigational consultant to the East India Company, described a mathematical way to construct the Mercator map which produced a better approximation than the original (Sriraman, Roscoe & English, 2010).

<sup>ii</sup> Math 606 was a topics course in the history of mathematics taught by Professor Bharath Sriraman in Spring 2009. One of the assignments in the course was to take Carslaw’s (1924) paper and rewrite in such a way as to make it readable by modern students. In an effort to get more mileage out of the work to be done, the students asked if it would be possible to turn the assignment into something that could be used in the future – namely an activity designed for use in a calculus classroom. Dr. Sriraman allowed for the change and arranged for the activity to be completed by undergraduate students who had completed Honors Calculus II. He also encouraged us to use the opportunity to perform some research.

## **-Guest Editorial-**

### **A Starting Point**

*Ke Wu Norman<sup>1</sup>*  
*Dept of Mathematical Sciences*  
*The University of Montana*

As a new faculty member in the Department of Mathematical Sciences at the University since Fall 2008, I was eager to find possible research partners with whom to build a research network for long term collaboration and support. I wondered “Where shall I start?” and “Whom shall I invite?” After several conversations with my colleague, Dr. Bharath Sriraman, and many phone calls to my good friend, Dr. Anne Kern, a new faculty at the University of Idaho, we came up with the idea of forming a support group of female faculty in the fields of mathematics and science education in the northwestern region of the United States, including Montana, Idaho, Oregon, and Washington. These four states face similar challenges in education such as a large number of K-12 public schools with small enrollments located in rural areas.

With funding support from the PACE program<sup>2</sup> at the University of Montana, a group of seven women researchers came to Missoula for a two-day meeting in the late summer of 2009. Participants included Drs. Anne Adams (University of Idaho, Moscow), Elizabeth Burroughs and Jennifer Luebeck (Montana State University), Anne Kern (University of Idaho, Coeur d’Alene), Libby Knott (Washington State University, Pullman), Min Li (University of Washington, Seattle), and Jerine Pegg (University of Idaho, Moscow then, and now University of Alberta, CA).

A wide range of topics were discussed during the meeting: (1) How to interest high school females in college science and mathematics majors; (2) The discourse of problem posing and problem solving; (3) Curriculum design that integrates engineering into mathematics and science coursework for K-12 Teachers; (4) Experiences in middle school mathematics lesson study; (5) Assessment: Examining how students from different groups interpret test items; (6) Rehearsal or reorganization: two patterns of literacy strategy use; and (7) The influence of a multidisciplinary scientific research experience on teachers views of the nature of science. Hopefully this is just a starting point among the women researchers in mathematics and science education from the northwestern U.S. region.

---

<sup>1</sup> Ke.Norman@mso.umt.edu

<sup>2</sup> Partnership for Comprehensive Equity

The papers in this section reflect the ongoing efforts of the participants and themes discussed in the meeting. Given that last year's meeting was a success, we are planning on continuing to build our professional collaboration again this year at the University of Montana<sup>3</sup>.

---

<sup>3</sup> We are grateful for the funding support provided by The University of Montana and the intellectual provided by the TMME editor, Dr. Bharath Sriraman

## **Rehearsal or Reorganization Two Patterns of Literacy Strategy Use in Secondary Mathematics Classes**

*Anne Adams<sup>1</sup>*  
*University of Idaho*

*Abstract: This study presents two critical cases illustrating distinct patterns in teachers' use of literacy strategies in secondary mathematics classes. The cases are part of a professional development project designed to enhance teachers' pedagogical skills by developing content literacy strategies for use in secondary mathematics and science classrooms. Teachers' beliefs about teaching mathematics, their uses of writing and vocabulary development strategies, and goals for student learning were examined via interviews, classroom observations, reflections on teaching, and teacher posts to an on-line discussion forum. Results show patterns of literacy strategy use were related to teachers' views of pedagogy and of mathematics. Ned, who held a procedural approach to teaching mathematics, used strategies as a rehearsal tool to support remembering correct ideas, fact, and procedures. Christine, with a conceptual approach to teaching, used literacy strategies as a tool to support deepening and reorganizing student understanding of mathematical concepts and relationships.*

**Keywords:** literacy strategies; mathematics teacher education; mathematics and writing; teacher professional development

Language is the medium of human interaction as well as much of human thought. As such, learning mathematics is as much about learning language as about mathematical objects and relationships. The language of mathematics both describes concepts and helps to shape them (Usiskin, 1996). "Words are tools for thinking—in mathematics as well as in other disciplines" (Countryman, 1992), p. 57). Mathematicians and students use language to make sense of new information, develop new ideas, and organize their understanding of the relationships among these, as well as communicate their understanding. Essentially, the use of language is integrally involved in the development of concepts and relationships and in our understanding of the world around us (Vygotsky, 1962). Without language we would not have mathematics.

---

<sup>1</sup>College of Education; Department of Curriculum & Instruction  
P.O. Box 443082  
Moscow, Idaho 83844-3082  
aeadams@uidaho.edu

### *Literacy: Tools for Thinking about Mathematics*

Effective use of literacy skills underlies effective use of mathematics. “Language arts provide the tools for teachers and students to read and understand problems, to write and draw their way toward understanding, and to communicate effectively” (Fogelberg, Skalinder, Satz, Hiller, Bernstein, & Vitantonio, 2008, p. 4). One aim of reform based mathematics education is to create classrooms where mathematical understanding is a reality for students. Infusing literacy strategies into instruction may provide a key. The National Council of Teachers of Mathematics (NCTM) argues the importance of giving students “experiences that help them appreciate the power and precision of mathematical language” (NCTM, 2000, p. 63). When writing to learn, different from writing to demonstrate learning, ideas occur as one writes. This practice offers opportunities to practice a variety of mathematical and thinking processes including observing patterns and relationships, making generalizations and conjectures, inferring, predicting, communicating, summarizing, interpreting, organizing, explaining, representing mathematical ideas, reflecting, and justifying one’s thinking. These are components of NCTM’s process standards for learning mathematics described in *Principles and Standards for School Mathematics* (NCTM, 2000). Students should be able to communicate their mathematical thinking coherently and clearly, analyze and evaluate the mathematical thinking strategies of others, and use mathematical language to express ideas precisely (NCTM, 2000).

With this information in mind, the Literacy Instruction for Secondary Mathematics and Science Teachers (LIMSST) project was created. The goal of this professional development project was to enhance the pedagogical skills of secondary mathematics and science teachers by infusing additional literacy skill instruction into their curriculum. The project set out to encourage and support them in developing content literacy strategies for use in their classrooms. The aim was to teach a selection of research based literacy strategies and to support the teachers over the course of a school year as they learned to integrate these strategies into their content area courses. Although the literature suggests many potential benefits of integrating literacy strategies into mathematics classes (Countryman, 1992; Forget, 2004; Murray, 2004), there has been little research showing how such strategies are actually being used. This study examines the literacy strategy practices of mathematics teachers in the project.

### *Literacy Tools for Learning*

Writing in mathematics can play a positive role in students’ construction of knowledge during mathematics learning activities. Writing serves as a means of helping students organize, analyze, interpret, and communicate mathematical ideas, leading to a deeper understanding of content concepts (Burns, 2004; Holliday, Yore, & Alverman, 1994). “Writing in mathematics can also help students consolidate their thinking because it requires them to reflect on their work and clarify their thoughts about the ideas” (NCTM, 2000, p. 61). The writing process can play a vital role in developing mathematical literacy and understanding. Research has shown that writing increases understanding, achievement, and problem solving skill (Bangert-Drowns, Murley, & Wilkinson, 2004; Borasi & Rose, 1989; Clarke, Waywood, & Stephens, 1993; Herrick, 2005; Steele, 2005, 2007).

The language of mathematics is both abstract and complex. In order to help students work through the complexities, they need experiences connecting everyday language to mathematics, distinguishing the various meanings and contexts of mathematical vocabulary terms, and connecting new mathematical knowledge with prior knowledge (Harmon, Hedrick, & Wood, 2005; Rubenstein, 2007). Frequent opportunities to use mathematics in context serve to create connections between mathematical ideas and how they are used, just as foreign languages are learned through use in context.

Beneficial learning opportunities include higher order processing skills such as analysis, synthesis, and evaluation of concepts and definitions (Adams, Thangata, & King, 2005; Monroe, 1997). In general, students need support in learning to use specific text features and in reading comprehension strategies as they interact with text and interpret the meaning on the page (Carter, & Dean, 2006; Kenney, Hancewicz, Heuer, Metsisto, & Tuttle, 2005; Vacaretu, 2008). Through this process, students of mathematics develop skill in reading mathematical text and construct meaning of mathematical ideas. Reading and writing can serve as tools for learning and thinking about the language and concepts of mathematics. Use of writing as a meaning making tool in mathematics class can support students both in learning mathematics and developing higher order thinking skills.

### *Theoretical Framework*

The meaning that one constructs for mathematical ideas is interlaced with the language with which one learned to reason about those concepts, words, and symbols (Pimm, 1995). Effective vocabulary instruction supports students in their own sense making by connecting words and concepts to students' prior knowledge, involving students in higher order thinking as they develop meaning for words, and providing frequent opportunities for them to use mathematical vocabulary and language in meaningful ways (Adams, 2003; Adams, Thangata, & King, 2005; Harmon, Hedrick, & Wood, 2005; Monroe, 1997; Thompson & Rubenstein, 2000). Constructing visual representations of concepts and vocabulary not only serves to deepen understanding, but to improve ability to recall the knowledge for later use (Marzano, Pickering, & Pollock, 2001; Rubenstein & Thompson, 2002).

Noting students' difficulties in understanding formal definitions for mathematical concepts, (Tall & Vinner, 1981) distinguished between the constructs of concept image and concept definition in explaining individuals' conceptions of concepts. They termed the concept definition as the formal words used to define a concept. In contrast, the concept image consists of all the various mental images, processes, and any associated properties brought to mind by an individual in considering a given concept. This concept image denotes an individual's conceptual structure of a concept (Tall, 1988; Tall & Vinner, 1981). When thinking of a term, it is the concept image that comes to mind, not the concept definition. The various components of an individual's concept image need not be coherent or consistent. A learner's concept images may also be in conflict with the formal concept definition accepted by the mathematics community. Concept images are based on an individual's experiences and may embody many facets not included in the concept definition. For example, the concept *slope of a line* is defined as rise/run, yet the term *slope* may evoke in an individual the various images of a graph of a line and its associated steepness, a table of values indicating the relative change in y for a one unit change in x, a ski slope or a road (which may have a changing slope), a formula for calculating the slope of a line, or the coefficient of x in a linear equation. Tall (1988) contends that students cannot use



concept definitions alone. When a student is simply given a concept definition, it forms a weak concept image. Since humans rely primarily on their concept images, students need opportunities to develop strong concept images that are in accordance with the desired concept definition.

### *Two Views of Vocabulary Development*

Two broad views of vocabulary development are presented. Based on constructs identified by Tall and Vinner (1981), I have termed these the concept definition view and the concept image view.

*Learn the Concept Definition.* This view holds that each vocabulary term has a precise and static verbal definition that must be learned. Such definitions are found in mathematics textbooks or in dictionaries and represent the true meaning of the term. Meanings are fixed and external to students, who all need to learn the same meaning for each term. Knowledge of the correct definition should provide an individual with a correct concept image. Useful learning activities make terms and definitions memorable or allow one to review the terms so they are not easily forgotten. Common approaches include presenting formal definitions for terms or asking students to find these in textbooks or dictionaries.

*Develop Concept Images.* An alternative to the view above is that of developing concept images. In this view, students make sense of each term in their own personal way. Experiences with a concept lead learners to associate various mental images, processes, and characteristics with that concept. These various associations comprise the learners' images of the concept. Learners construct concept definitions from these concept images. While the terms that label concepts were created by other humans and carry commonly understood meanings, the sense that individual students make of these meanings is highly personal (Tall, 1988; Tall & Vinner, 1981). Such meanings take time to develop and are best approached in multiple ways, exploring critical attributes of concepts and examining relationships between terms and ways to connect meanings to other aspects of one's life or of mathematics. Through a variety of appropriate experiences, concept images become more refined as understanding of the concept deepens, and moves closer to the formal concept definition (Fogelberg et al., 2008; Marzano, 2004; Tall, 1988).

### *Two Views of Writing*

Two distinct views of writing, a product centered view and a process centered "writing to learn" view, have emerged in education. Within the product centered view of writing, the purpose of writing is to create a polished piece of work that demonstrates what the writer knows. This work should be factually and grammatically correct (National Writing Project & Nagin, 2003). Such writing records what has been learned and can be viewed as "writing to record." The process view of writing takes a different approach. In this view, students learn through the process of writing. Such writing can support thinking and learning as the writer analyzes, interprets, and synthesizes ideas, constructing new understandings (Emig, 1977; Forget, 2004), and can be viewed as writing to learn. The resulting writing may be personal and unpolished, but presents a record of the learner's thinking while attempting to understanding new concepts and relationships or to solve problems.

*Writing to record.* Teaching within the product view of writing focuses on recording accurate, factually correct content. This content may have been memorized as accuracy is critical and texts and teachers are likely to have more accurate information than learners. The polished, final product of such writing serves as a record of what has been learned and may be assessed for errors.

*Writing to learn.* The process or "writing to learn" view focuses on the process of developing and explaining ideas. Within this view, writing is learned through the process of writing and is fundamental to learning in all content areas. In writing, students analyze, synthesize, and interpret content, thereby constructing new knowledge (Emig, 1977; National Writing Project & Nagin, 2003). Writing is a powerful reasoning tool itself and serves to make thinking visible.

To illustrate, if a teacher asks students to write the steps for finding a per cent equivalent to a fraction, the teacher is exhibiting a "writing to record" view of writing. A teacher who asks students to write about connections between fractions and percents, or asks them to write about the similarities and differences between a fraction and its equivalent per cent is demonstrating a "writing to learn" view. Students are asked to think about a new idea or connection and write about it as they think. This view is also illustrated in asking students to explain why a procedure works or justify a solution to a problem.

#### *Literacy Instruction in Math and Science for Secondary Teachers (LIMSST)*

The LIMSST project was developed and implemented by three faculty members in the College of Education at a public university, one in science education, one in language arts education, and myself in mathematics education. Funding was awarded to the university and six partner rural school districts (five of them identified as high-need districts) in a northwestern state for the academic year 2007-2008 by a Federal Eligible Partnership Subgrant. Key project activities included a one week summer workshop to develop literacy strategy use. On-going support throughout the year was provided via three follow-on workshops, three classroom visits to observe and support teachers in literacy strategy use, and a learning community linked via required participation in an on-line discussion forum. Throughout the teacher workshops, project staff actively used the same literacy strategies they were teaching participants to use. Participants worked and learned collaboratively, focusing deeply on how learners use language to make meaning of content and on using strategies of reading, writing, and oral discussion to do so.

The following broad themes were developed on multiple levels throughout the year across all project activities:

- Learning involves making meaning of information and one's experiences.
- Literacy tools for learning and thinking involve reading, writing, oral discussion (speaking and listening), and thinking (reflection).
- Literacy strategies can be integrated into instruction as meaning making tools.

Project staff presence at the schools for observations and conversations was important both to serve as a reminder to integrate literacy strategies and to demonstrate our interest in helping teachers learn how to do this effectively. During the visits, staff were able to use the observations as starting points for conversations about how each teacher could modify existing practices to support the development of student thinking about mathematics using literacy strategies.

#### *Method*

The purpose of this study was to understand how mathematics teachers infused the literacy strategies developed through participation in a professional development project into their secondary mathematics classes. This qualitative case study used a constructivist perspective to examine the nature of literacy strategy use by 12 mathematics teachers who participated in the Literacy Instruction for Secondary Math and Science Teachers (LIMSST) professional

development project. The study examined two questions: 1) What literacy strategies did the LIMSST teachers use in their secondary mathematics classes and how were they using and/or modifying literacy strategies? and 2) What influenced teachers' use of literacy strategies?

#### *Participants*

The LIMSST project participants consisted of a group of 15 secondary teachers (12 mathematics, 3 science) who shared the intent to learn and infuse literacy strategies into their classes over the course of an academic year. Because research has shown that few mathematics teachers use content literacy strategies (Fisher & Ivey, 2005; Lesley, 2005; Moje, 2006), these teachers provided an opportunity to observe how secondary teachers use literacy strategies in mathematics instruction. This study is limited to the 12 mathematics teachers in the project.

The project teachers were from small, often isolated, rural communities. These teachers had the benefit of small classes, ranging from 4 to 20 students. However, in some cases this benefit was offset by the need to teach as many as six different courses or subjects in a day and by a lack of mathematics teaching colleagues at their grade level (or at any grade level) with whom to collaborate. Many of these districts have found meeting state mandated goals challenging due to limited funding and a variety of social problems facing the communities. The 12 mathematics teacher participants had a variety of backgrounds, perspectives, and teaching histories. All had volunteered for the project and had some level of administrative support for participating; some had stronger administrative encouragement to participate. They were paid a small stipend, primarily for their time in preparing documents for the project. Nine of the teachers taught mathematics exclusively, one taught both mathematics and science, and two taught all subjects in self-contained classrooms, one in 6<sup>th</sup> grade and the other in a multi-grade class with eight students spanning grades five through eight. Three were junior high teachers, four were high school teachers, and three taught both junior and senior high school classes.

#### *Data Collection*

The findings of this study are based on data gathered in the LIMSST professional development project over the course of the 2007-2008 academic year. Data sources include formal interviews with participant teachers, classroom observations and observation debriefs, teacher posts to an on-line discussion forum, planning and participation notes from professional development workshops, and artifacts such as lesson plans using literacy strategies, teachers' reflections on teaching these lessons, and examples of student work. Participants completed the Teacher Belief Inventory (Luft & Roehrig, 2007), a seven item protocol designed to elicit teachers' beliefs about teaching, learning, and students in mathematics and science classes. In addition, teachers answered four questions probing their beliefs about content literacy strategy use in mathematics and science classes and their purposes in using reading, writing, vocabulary development and discussion strategies in their content classes.

#### *Data Analysis*

Data were analyzed and interpreted through the lens of the role of literacy in developing mathematical understanding. The data analysis process began with ongoing preliminary analysis (Ely, Vinz, Downing, & Anzul, 1997), in which frequent reading and categorizing of data served to inform further data collection, to shape categories, and to consider potential themes. Once data collection was completed, summative analysis was conducted to identify essential themes and features of the data (Ely et al. 1997). This involved further consolidation and interpretation of the data.

Data were analyzed from three perspectives: by teacher, by literacy strategy, and by the analytic themes that emerged from preliminary analysis. The themes included which strategies teachers implemented, how teachers used strategies, teachers' purpose in using a strategy, and how teachers think about teaching mathematics. The intent was not to describe use of specific strategies by individual teachers, but to determine the variety of ways in which literacy strategies might be used across mathematics teaching and to investigate factors influencing such use.

In looking for patterns of strategy use and the meanings teachers brought to strategy use, increasing attention was paid to how teachers introduced and framed each literacy strategy for students, clues about teachers' purpose in using a particular strategy, and what a teacher hoped students would gain from use of the strategy. Periodic discussion with project staff and ongoing reading of related literature additionally continued to shape understanding of the data and the relationships and influences on teachers' use of literacy strategies.

Upon further examination of the data, it became evident that teachers' patterns of literacy strategy use cut across writing, reading and vocabulary development strategies and were aligned with teachers' learning goals for their students. An analytic framework was developed from the data and supported by literature. This framework presents two general patterns of literacy strategy use. I have termed these patterns the Rehearsal pattern and the Reorganization pattern.

Trustworthiness was addressed via multiples sources of data, thick description, and a lengthy time in the field. Data collection took place over a period of more than a year, beginning in May 2007 and ending in June 2008. During this time, a large amount of data was collected from multiple sources. Teachers submitted 48 lesson plans and accompanying reflections, and posted more than 120 entries on the project's interactive website. In addition, 34 observations and 67 interviews were conducted. Member checking occurred at multiple points over the year as teachers met with me and other project staff, participated in workshops, and communicated via email and the interactive website. Additional checking took place in frequent discussions with other project staff.

### *Findings: Two General Patterns of Literacy Strategy Use*

Examination of literacy strategy use by all 12 teachers indicated two distinct patterns of use, rehearsal and reorganization. These patterns of use were exhibited primarily across the broad categories of vocabulary development and writing. The more prevalent pattern among project teachers was the rehearsal pattern of use. While there also appeared to be differences in the ways teachers used reading strategies, in general, reading was used minimally in mathematics class. As a result, the present discussion of strategy use will focus on vocabulary development and writing. After a general description of the Rehearsal pattern and the Reorganization pattern, illustration will be provided with two critical cases. Table 1 presents a summary of the two patterns.

#### *Rehearsal*

Teachers exhibiting the rehearsal pattern of literacy strategy used these strategies as tools to provide students with multiple opportunities to revisit, review, and rehearse facts, concepts, and procedures that had been formally taught. In the realm of vocabulary development, these teachers expected their students to learn and remember formal concept definitions for important terms. Writing was used as an additional opportunity to reexamine information. Students might be asked to review notes, class presentations, or sections of the textbook and summarize important ideas or procedures in their own words. If reading was assigned, it was typically

confined to reading word problems or to reviewing a textbook presentation of material that often had been previously taught in class.

#### *Reorganization*

With the reorganization pattern of literacy strategy use, teachers used strategies as thinking tools to support students in developing conceptual understanding of mathematical ideas and procedures. Vocabulary development strategies were used to help students develop concept images that they could use in making meaning of important terms and ideas. Writing was used as a thinking tool, to help students form and become aware of their own ideas about the mathematics concepts and procedures they were learning. Reading, while seldom used, was most frequently seen as a tool to help in interpreting word problems that students needed to solve.

Table 1: Two Patterns of Literacy Strategy Use

	<b>Rehearsal</b>	<b>Reorganization</b>
<b>Vocabulary Development</b>	Learn Concept Definition	Develop Concept Images
<b>Writing</b>	Record Procedures, Facts, Rules	Form Thoughts Raise Awareness of Own Ideas

#### *Ned and Christine: Critical Cases Exemplifying Two Ways of using Literacy Strategies*

Ned and Christine were the least experienced teachers in the LIMSST project. These two teachers shared many similarities in their backgrounds and in the classes they chose to target in developing literacy strategy use. Their background characteristics are summarized in Table 2. Although Ned was a generation older than Christine, both were reasonably new teachers, each with two years experience teaching high school mathematics. Both had earned master's degrees prior to teaching high school and had taught mathematics to older students before earning a teaching certificate. Christine had earned a standard secondary mathematics teaching certificate, while Ned had followed an alternative route to certification.

Table 2: Background characteristics of Ned and Christine

	<b>Ned – Rehearsal</b>	<b>Christine - Reorganization</b>
Age	early 50's	early 30's
Education	M.S.	M.A.T.
K-12 Teaching Experience	2 years high school teaching, mathematics	2 years high school teaching, mathematics and Spanish
Target Class	Algebra 1 Lower achieving students	Integrated Math ( topics in Algebra 1 and Geometry) Lower achieving students
Previous Teaching Experience	Taught math to adults night school – 1 year	Taught math at community college – 2 years Tutored math at university center – 3 years
Prior Background	Engineering/military	Accounting
Certification	Alternative Secondary Mathematics Certificate	Standard Secondary Mathematics Certificate

Despite their many similarities, these two teachers exhibited distinctly different ways of using literacy strategies and had very different purposes in their uses of these strategies. The Rehearsal pattern of use was exemplified by Ned and the Reorganization pattern of use was shown by Christine. These teachers' patterns of use are compared in the following section.

### *Two ways to Use Vocabulary Development Strategies*

The LIMSST project workshops had introduced a variety of vocabulary development strategies that could be used to support students in their own sense making of concepts and terms. These strategies included the Frayer Model, word sort (Barton and Heidema 2002), and word wall (Fogelberg et al., 2008). Project teachers themselves later introduced the Visual Verbal Word Association (VVWA), a strategy similar to the Frayer Model. In general, project teachers chose a few vocabulary development strategies to implement and used these in ways consistent with their views of vocabulary and concept development and with their previous teaching practice. All teachers tended to use the same few vocabulary development strategies; however, their views of vocabulary development led them to use the strategies in two different ways. Hence, despite using the same vocabulary strategies, students in these classes were offered different types of opportunities to learn the language of mathematics, based on their teacher's view of vocabulary development.

Teachers who viewed vocabulary as a set of concept definitions to be learned provided learning experiences intended to enhance memory of terms and their definitions. Teachers who focused on supporting students in developing rich concept images used the same strategies as the other teachers, but used them to help students make meaning of concepts, develop multiple rich concept images, refine these images, and examine relationships between concepts.

The Frayer Model (Barton & Heidema, 2002; Billmeyer & Barton, 1998; Roe, Stoodt, & Burns, 2001) proved to be a favorite tool among project teachers. This tool uses a graphic format to help students develop their understanding of concepts and conceptual relationships and understand what the concept is and is not. Students complete a diagram with a definition for the term in their own words, determine the term's essential attributes, and refine their understanding by selecting examples and non-examples of the concept from their own experiences. Using this strategy, students essentially explain their own understanding of a concept. An example of the Frayer model is presented in Figure 1.

The Verbal Visual Word Association (VVWA) is a similar strategy commonly used by project teachers. It is also organized within a graphic format and includes a definition, a visual representation of the term, and characteristics or some personal association with the term (Barton & Heidema, 2002). VVWA is useful for concepts that have a visual component or are more concrete, such as geometric figures, or that show a relationship, such as the slope of a line which relates the rise to the run on a graph. Figure 2 presents an example of VVWA from Christine's class.

### *Vocabulary Development Strategies Used for Rehearsal: Ned*

Ned's approach to vocabulary development was to emphasize learning and remembering the formal concept definition. His classroom activities used vocabulary development strategies in ways that made formal definitions visible and provided opportunities to practice and review these. Ned looked for evidence that students knew the formal definition of vocabulary terms. "Make sure that they have the definition. Make sure that they have, in their notebooks, they write down the vocabulary....The Frayer Model is a good one to relate those—example and definition

and diagram" (Ned, TBI Interview, 5/21/08). In Ned's view, each of these aspects serves to make the true meaning apparent to the learner and, once recorded, is available for reference and review.

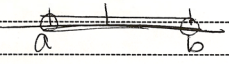
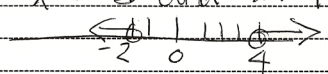
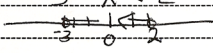
Definition (in your own words)	Facts/Characteristics
A solution of a compound inequality joined by 'and' and makes both inequality true	$a < x < b$
"2 inequalities joined together with and"	$x > a \text{ and } x < b$
	
Compound Inequalities that are joined with the word AND.	
Examples	Non-Examples
$-3 > x \text{ and } x < 2$	$x < -3 \text{ and } x > 4$
$-3 < x < 2$	
	

Figure 1: Frayer Model – Student work from Ned's class – 11-07-07

In using the Frayer Model (Figure 1) as a tool to teach a concept definition, students must understand the definitions they are given and revisit these often, trying to memorize the words. They may not have had opportunity to connect the definition with any personal meaning, with concept images, or with other mathematical ideas. Hence, the only way to learn and remember these is to revisit and rehearse the information often. The Frayer Model provides a convenient way to do this, because it organizes useful information about the term.

It was important to Ned that students have the correct information, and he did not expect them to figure it out for themselves. "We worked on Frayer Models for the same vocabulary words but they had to use the definitions from their notes" (Ned, Web Post, 12/12/08). In using the Frayer Model, Ned provided students with all the information he wanted them to include. At times, he allowed students to record information directly onto the Frayer model form as he lectured. "I give an answer for each of the four sections of the Frayer Model during my lecture. I have a stack of Frayer Model blanks available. I normally pass them out when Vocab words are coming up. Sometimes I wait until after the lecture so they have to transcribe from their notes" (Ned, Web Post, 12/07/08).

During one observed lesson, Ned used the Frayer Model as a review before a test. He provided multiple blank Frayer Model forms. Each form named a concept related to inequalities or absolute value equations and provided an entire page on which to write. Ned's goal was for students to reread information in the text or in student notes, providing another opportunity to learn the material they had been given. Ned wrote that the activity "was a restatement and summary of notes they should have had in their math notebooks...[It] required the students either

to read the section in the book again or the review pages. Some of the students actually found information in their notebooks” (Ned, Reflection, 11/07/07).

For Ned’s students the task was to find and learn the concept definition, as opposed to generating their own. When used in this way, the Frayer Model is transformed from a tool for student use in developing and refining their concept images into a device for recording the concept definition and illustrative examples.

*Vocabulary Development Strategies Used for Reorganization: Christine*

Christine designed learning activities to help her students develop concept images, the various pictures and images one draws on when thinking of a concept or term. Her use of the Frayer model illustrates its application as a tool to support students in development of their concept images. “I actually conducted a dice throwing experiment to show the difference between [theoretical and experimental probability]....After we showed this, I just let them do the Frayer model based on what they learned from the modeling” (Christine, Web Post, 11/20/07). Completing the Frayer model helped students organize their thinking about the concept. Christine described ideas for scaffolding students in finding and thinking about relevant information to use in completing a Frayer model. “You could ask them what they know about a particular topic, write down anything they say on the board and then they can use that list to help them to fill out the chart [Frayer model diagram]” (Christine, Web Post, 11/20/07).

Christine tried using the VVWA strategy to help students differentiate perimeter, area, and volume: “They drew pictures for them. On the VVWA's, for the characteristics of the shapes, they wrote down the formulas for area. I tried to avoid giving them formulas for perimeter, since they really aren't necessary.” When her students continued to confuse the terms *area* and *perimeter* she “had them daily write definitions in some format for area and perimeter, until it was clear they could tell the difference” (Christine, Web Post, 11/29/07).

Practice using terms has a role in a meaning making approach, but this role is different than in a concept definition approach. While the meaning of a concept or term must be constructed, the label must be learned and connected to that meaning. In Christine's classroom, what students were practicing was making meaning of these terms, not a formal definition. Practice was involved in creating these meaningful connections, but it was characterized by engagement in a variety of related experiences, rather than rehearsal of information.

In February Christine again tried the VVWA strategy, after engaging students in an activity to develop meaning for midpoint of a line segment. Reflecting on the lesson, she wrote:

The VVWA went well. They developed their own meaning, and it was correct, of the word and were able to reinforce this using the VVWA. I have taught this lesson before but instead of allowing them to discover the meaning of midpoint, I just told them about it. They really seemed to grasp the idea of midpoint much more than students in previous years did. (Christine, Reflection, 2/27/08)



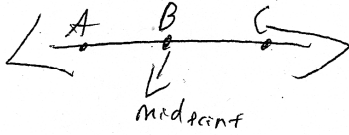
VWA	
<u>Vocabulary Term</u>	<u>Visual Representation</u>
Midpoint	
<u>Definition</u>	Personal association or characteristic (could be examples)
a point that divides the points $\overline{AC}$ into two congruent segments	Congruent points equal

Figure 2: Visual Verbal Word Association: Student work from Christine's class: 2-27-08

She later elaborated on her approach:

I guide them quite a bit, but we usually do some sort of activity where they "discover" the concept. For example, with area, they have to cover squares and rectangles with centimeter grid paper, count the number of unit squares that cover the surface (to get the idea of area), and then from there they develop the formula for area of rectangles and squares. But by the end, most can tell me in their own words what the definition of area is. Then we fill out the VWA, but I ask them to tell me what a good definition would be. (Christine, Web Post, 12/08/08)

Christine believed that ideas about a concept must be developed. Words can help with this, but are not enough. Learners also need experiences and examples of concept in order to make meaning of them. Christine's students were encouraged to share ideas and use reference materials as they made meaning of a concept. She assessed their current understanding frequently and continued to provide new learning activities, giving them time to revisit concepts as they refined their understanding and continued to create additional elements of concept images. Time and a long list of topics to teach limited Christine in this. She could not provide such time for every concept, so chose the central concepts that would receive emphasis. Her goal was to support students in making meaning.

In summary, Ned and Christine used the same vocabulary development strategies, but had different purposes in using them. While Ned gave his students frequent opportunities to learn and rehearse formal concept definitions for terms, Christine's students engaged in multiple activities intended to make meaning of concepts, develop multiple rich concept images, refine these images, and examine relationships between concepts.

### Two Ways to Use Writing

LIMSST teachers found frequent opportunities to engage their students in writing about mathematics. But in this area too, strategies were used in different ways. Some teachers described the writing process as helpful in remembering information, while others used writing to help students think or to help them connect their current knowledge to new ideas.

#### Writing as Rehearsal: Ned

Ned used lengthy writing assignments that required students to supply rules and examples for various procedures he had taught. The purpose of these assignments was two-fold: to review and remember procedures and definitions, and to focus students on topics for review. Ned said, "I think that being able to describe what you are doing mathematically in words is helping you retain it more" (Ned, TBI Interview, 5/21/08).

For example, in a lesson to review methods of solving inequalities, Ned first asked students to read a review section in the text. He then asked them to use their own words to explain how to solve inequalities by adding or subtracting, and to provide an example. Students were also asked to write how to solve inequalities by multiplying or dividing by a positive number and by a negative number as well as to provide related definitions and examples (Ned, Observation, 11/07/07). Samples of student writing from this lesson (Figure 3) are focused on steps of procedures and some sections are incomplete. A number of student papers show the same phrases or examples, suggesting that these appeared in the text or in class presentations.

Student C		Student D	
Explain in your own words how you ...	Give an Example	Explain in your own words how you ...	Give an Example
Solve Inequalities by adding or subtracting. on the variable side! If adding subtract that # from both sides If subtracting, add that # to both sides	$\begin{array}{r} x-3 > 11 \\ +3 \quad +3 \\ \hline x > 14 \end{array}$ $\begin{array}{r} x+5 < 9 \\ -5 \quad -5 \\ \hline x < 4 \end{array}$	Solve Inequalities by adding or subtracting. add or subtract on <u>both</u> sides	<del>17KAT-19A</del> $\begin{array}{r} 2 < x-1 \\ +1 \quad +1 \\ \hline 3 < x \end{array}$
Solve Inequalities by Multiplying or Dividing by a Positive Number. on the variable side! If mult, divide by that # on both sides If div, mult by that # on both sides	$\begin{array}{r} 3m > 9 \\ \div 3 \quad \div 3 \\ \hline m > 3 \end{array}$ $\begin{array}{r} 4, \frac{m}{4} < 16 \cdot 4 \\ \hline m < 64 \end{array}$	Solve Inequalities by Multiplying or Dividing by a Positive Number. You would multiply or divide to both sides.	<del>17KAT-19A</del>
Solve Inequalities by Multiplying or Dividing by a Negative Number. Same as with a positive number, except change the inequality sign and the answer to a negative or positive	$\begin{array}{r} -2m > 10 \\ \div -2 \quad \div -2 \\ \hline m < -5 \end{array}$ $\begin{array}{r} \frac{m}{5} < -3 \cdot 5 \\ \hline m > 15 \end{array}$	Solve Inequalities by Multiplying or Dividing by a Negative Number. multi. or divide by the negative # on both sides. Remember to change your sides.	

Figure 3: Sample of student writing from Ned's review of methods of solving inequalities

*Writing as Reorganization: Christine*

In contrast to Ned, Christine used short writing activities that were integrated throughout a learning activity with many related components. Her purpose in choosing these activities was to support student thinking and meaning making, with a goal of students developing or reorganizing a set of related understandings.

In Christine's view, writing about one's process helps both with remembering and with awareness, bringing one's learning more clearly into focus. But it also serves as a reflection tool as students examine and explain their own thinking. She asked students to "write to make meaning of terms and concepts. Using kids' meaning makes much more sense to them" (Christine, Interview, 2/27/08). Other writing focused students thinking on their own background knowledge in preparation for connecting it to new learning. She explained:

I use writing quite a bit more than I use reading....I wanted them to think about what it was they were doing and explain....I had them explain their steps to problems. I had them write journals about things that they learned or everything they know about a particular topic. Sometimes I would have them write about something before we talked about it. "What's your prior knowledge of this topic?" (Christine, TBI Interview, 5/29/08)

When her students wrote about a given topic, Christine did not expect correct descriptions of information that had been taught. She was instead looking for descriptions and explanations of students' current understandings of the topic and of its relationships with various mathematics concepts. For example, in her lesson about line segments and midpoints, Christine began by asking students to engage in an activity to help develop meaning of *midpoint* before they completed the VVWA shown previously. Ensuing work engaged students in using the term in several short problems and writing their thoughts as they proceeded. For example, they were given the statement, "B is the midpoint of AC" and asked to "write what this tells you about the segments and about the lengths of the segments." This exercise was followed by a similar one: "AB = 2. Write what other information you can conclude about these segments." (Christine, Observation, 2/27/08) The lesson concluded with students making a brief open-ended journal entry in which students are asked to write what they have learned about segments and their lengths (Figure 4). Writing samples from Christine's class vary widely, both in ideas presented and in words and examples used to portray these ideas. Her students write in personal ways describing the sense they are making of ideas.

Student A:

i have learned that the capital letters AB  
w/out a line over them mean the length  
of the segment  $\overline{AB}$ . I have also learned  
that the midpoint, for example if you  
have  $P \quad Q \quad R$  then Q is the  
midpoint if it divides

Student B

LL#3 : 2-27-08  
What you have learned about segments and their lengths.  
I have learned that a segment is a line consisting  
of two end points. It also can contain some points  
in the middle.

Figure 4: Samples of student writing from Christine's midpoint lesson.

### *Two Purposes of Writing in Math*

While Ned and Christine both engaged students in mathematical writing, their uses of writing served very different purposes. Ned used writing as a memory tool. In writing, his students created a personal reference tool that summarized material that had been taught in class. Such writing focused on recording what students had learned, such as facts, definitions, rules, or procedures. It might also record steps students used in their own execution of a procedure. Writing served to raise student awareness of what they had learned and resulted in a written record that could be used for review. Ned could also use this record as an assessment tool to determine what students have remembered.

In contrast, Christine used writing as a student thinking tool to help students construct their own understanding of concepts and relationships or to solve problems. She encouraged students to describe their current understanding of concepts, relationships, or strategies, or to analyze patterns and relationships. She also encouraged them to make connections between their current knowledge and new ideas. Such writing could also be used for assessment of the nature of student understanding. Reading student writing allowed Christine to make purposeful instructional decisions and to design learning activities that improved or deepened student understanding.

### *Teachers' View of Learning Mathematics*

In addition to differences in how literacy strategies were used, Ned and Christine differed in their views of their role as a teacher. Ned had a procedural focus to teaching mathematics, and

he viewed his role as a teacher as one of “guiding the students through a process so that they can obtain those skills” (Ned, TBI Interview, July 2007). His view of mathematics teaching and learning is in line with a traditional procedural view of mathematics.

Teachers with a traditional orientation view mathematics as having an existence independent from human existence, and learners passively receive this mathematical knowledge by listening to or watching knowledgeable others (Philipp, 2007; Simon & Tzur, 1999) and practicing procedures they have seen demonstrated (Smith, 1996). There is a fixed body of information to be learned and the teacher’s role is to transmit this information and ensure that students have received it (Barkatsas & Malone, 2005). This view is associated with a focus on mathematical procedures and correct answers, with the role of problem contexts being minimized (Thompson, 1992). It is also referred to as a procedural orientation.

In contrast, Christine's views were consistent with a conceptual orientation toward teaching and learning math. “I think my role is to help guide students to learn and not just to teach and tell, teach by telling, but trying to find ways to help them discover for themselves (Christine, TBI Interview, July, 2007). Within a constructivist or conceptual orientation teachers view knowledge of mathematics concepts, relationships and procedures as constructed by the student through intellectual engagement in mathematical exploration and problem solving, analyzing patterns and relationships, and justifying their mathematical reasoning. Learning is a process of meaning making on the part of the learner, and teaching is a process of facilitating students’ meaning making (Jones, 1997). Teachers need to identify current student understandings and provide learning opportunities to support students in building on and extending these (Barkatsas & Malone, 2005).

Both Ned and Christine used literacy strategies in ways consistent with their views of teaching and learning mathematics. Ned provided opportunities for his students to receive, review, and rehearse important information. He believed students need to learn and remember correct ideas, facts, and procedures. Christine thought a teacher's role is to provide opportunities for students to make their own meaning and develop conceptual understanding of the structure of mathematics, and she supported her students in doing so. Despite their different viewpoints, each found literacy strategies helpful in pursuing their learning goals for their students.

### *Discussion of Results*

Teachers fit new teaching ideas into their current understanding of teaching and learning. Their views about learning and understanding serve as frames, the “underlying structures of belief, perception, and appreciation” (Schon, 1995, p. 23) and shape all aspects of their teaching. In this study, teachers' views of learning and understanding mathematics proved a powerful influence both on which literacy strategies they chose to use and on how they used these strategies. These frames determined what teachers noticed and how they interpreted what was noticed. Additionally, the frames determined what it meant to know and understand mathematics as well as what should be known in mathematics: a set of procedures to be executed or a complex and rich network of meanings and relationships between concepts and a variety of ways to use these in thinking about mathematics and solving problems. These views influenced the nature of goals set for learning, how teachers thought about understanding mathematics, the nature of learning experiences offered to students, and what it means to be a successful student.

This study demonstrated that when students are asked to use literacy strategies as tools for rehearsal, the emphasis is on revisiting and understanding the ideas of others. Students may review or restate these ideas, but they remain another's ideas and the student's task is simply to

learn, remember, and use these. However, when literacy strategies are presented as tools for meaning making, learners are encouraged to interpret their experiences in meaningful ways. They have the opportunity to interact with a variety of ideas: some their own, some proposed by others. From this interaction there is the potential to integrate ideas and reorganize them into a new and meaningful structure. This process is at the heart of sense making. While not all students will engage with ideas in such meaningful ways, the opportunity and support for doing so exists, and literacy strategies support such opportunities.

Purely providing teachers with literacy strategies is not enough to allow them to use the strategies as intended. In general, the same vocabulary development strategies were used by LIMSST project teachers of both views. Teachers implemented a small number of vocabulary development strategies and the nature of this implementation was consistent with their views of vocabulary and concept development. Teachers' views of vocabulary development led them to use the strategies in two different ways. Students with teachers holding a concept definition view had multiple opportunities to learn and practice concept definitions, while students with teachers holding a concept image view were engaged in learning experiences to develop a variety of concept images related to the concept at hand. Hence, despite using the same vocabulary strategies, students in these classes were offered different types of opportunities to learn the language of mathematics, based on their teacher's view of vocabulary development.

Similarly, the type of learning that was supported in the use of writing was related to teachers' views of writing and their goals for learning mathematics. Teachers who viewed writing as a tool for practicing or demonstrating what has been learned asked students to write accounts of rules and procedures or of applications of these rules and procedures. They also supported students in ensuring that these accounts were accurate and were responsible for selecting and shaping the information students recorded.

In contrast, teachers who viewed writing as a tool for thinking provided a variety of opportunities for students to write about their observations and understandings, and to form connections about mathematical ideas. They supported students in deepening personal understanding through the use of writing and through integrating writing with other learning activities. These teachers used writing as a tool for thinking in three ways. It was initially used to engage students in thinking about prior knowledge, making current knowledge available for use as a starting point in considering new ideas. Writing was also used in subsequent learning activities as students explored and analyzed mathematical ideas, observed relationships, and wrote their thoughts to clarify, organize, and revisit them. In a final use of writing, students were asked to make sense of and explain their observations and analyses, justifying their reasoning. In these classes, students were supported in exploring ideas that did not always match those of the teacher. Writing was used as a thinking tool for reflection and interaction with ideas.

### *Implications and Conclusion*

Literacy strategies can enhance learning in two ways. They can be used to increase student opportunities to focus on and practice procedures, increasing awareness of these, and providing additional opportunities to rehearse material to be learned. Alternatively, the strategies can support students in making observations, identifying patterns and relationships, clarifying thinking, supporting higher order thinking skills such as reasoning, justification, synthesizing ideas, and constructing new meaning. While both types of learning can benefit students, each affords a different benefit.

We are just beginning to understand how teachers can and do make use of literacy strategies in mathematics classes. Future research is needed to further examine factors that affect how teachers use literacy strategies. Such research would also provide better understanding of how to teach and support effective use of literacy strategies, for use in both teacher training and professional development. Additional research examining the relationship between a teacher's views of teaching learning and their use of literacy strategies would help us understand this complex relationship and provide tools to better support teachers in effective use of literacy strategies. Limited research suggests that literacy strategies are powerful learning tools for mathematics as well as other content areas. Research into the relationship between student achievement and literacy strategy use would strengthen arguments that these tools are effective across content areas.

## References

- Adams, T. L. (2003). Reading mathematics: More than words can say. *The Reading Teacher*, 56(8), 786-795.
- Adams, T. L., Thangata, F., & King, C. (2005). "Weigh" to go! *Mathematics Teaching in the Middle School*, 10(9), 444-448.
- Bangert-Drowns, R. L., Murley, M. M., & Wilkinson, B. (2004). The effects of school-based writing-to-learn interventions on academic achievement: A meta-analysis. *Review of Educational Research*, 74(1), 29-58.
- Barton, M. L., & Heidema, C. (2002). *Teaching reading in mathematics* (2nd ed.). Aurora, CO: Mid-Continent Research for Education and Learning.
- Billmeyer, R., & Barton, M. L. (1998). *Teaching reading in the content areas: If not me, then who?* Denver, CO: Mid-Continent Research for Education and Learning.
- Borasi, R., & Rose, B. J. (1989). Journal writing and mathematics instruction. *Educational Studies in Mathematics*, 20, 347-365.
- Burns, M. (2004). *Writing in math class*. Sausalito, CA: Math Solutions Publications.
- Carter, T. A., & Dean, E. O. (2006). Mathematics intervention for grades 5-11: Teaching mathematics, reading, or both? *Reading Psychology*, 27, 127-146.
- Clarke, D. J., Waywood, A., & Stephens, M. (1993). Probing the structure of mathematical writing. *Educational Studies in Mathematics*, 25(3), 235-250.
- Countryman, J. (1992). *Writing to learn mathematics: Strategies that work*. Portsmouth, NH: Heinemann.
- Ely, M., Vinz, R., Downing, M., & Anzul, M. (1997). *On writing qualitative research*. London: Routledge Falmer.
- Emig, J. (1977). Writing as a mode of learning. *College Composition and Communication*, 28(2), 122-128.
- Fang, Z. (1996). A review of research on teacher beliefs and practices. *Educational Researcher*, 38, 47-65.
- Fisher, D., & Ivey, G. (2005). Literacy and language as learning in content area classes: A departure from "every teacher a teacher of reading". *Action in Teacher Education*, 27(2), 3-11.
- Fogelberg, E., Skalinder, C., Satz, P., Hiller, B., Bernstein, L., & Vitantonio, S. (2008). *Integrating literacy and math: Strategies for K-6 teachers*. New York: The Guildford Press.

- Forget, M. A. (2004). *Max teaching with reading and writing: Classroom activities for helping students learn new subject matter while acquiring literacy skills*. Victoria, BC: Trafford.
- Harmon, J. M., Hedrick, W. B., & Wood, K. (2005). Research on vocabulary instruction in the content areas: Implications for struggling readers. *Reading & Writing Quarterly*, 21, 261-280.
- Herrick, C. J. (2005). *Writing in the secondary mathematics classroom: Research and resources*. State University of New York College at Cortland, Cortland, NY.
- Holliday, W. G., Yore, L. D., & Alverman, D. E. (1994). The reading-science-learning-writing connection: Breakthroughs, barriers, and promises. *Journal of Research in Science Teaching*, 31, 877-893.
- Kenney, J. M., Hancewicz, E., Heuer, L., Metsisto, D., & Tuttle, C. L. (2005). *Literacy strategies for improving mathematics instruction*. Alexandria, VA: Association for Supervision and Curriculum Development.
- Lawson, A. E. (1989). A theory of instruction: Using the learning cycle to teach science concepts and thinking skills. *National Association for Research in Science Teaching Monograph*, 1.
- Lesley, M. (2005). Looking for critical literacy with postbaccalaureate content area literacy students. *Journal of Adolescent & Adult Literacy*, 48(4), 320-334.
- Lortie, D. (1975). *Schoolteacher*. Chicago: University of Chicago Press.
- Luft, J. A., & Roehrig, G. H. (2007). Capturing science teachers' epistemological beliefs: The development of the teacher beliefs interview. *Electronic Journal of Science Education*, 11(2), 38-63.
- Marzano, R. J. (2004). *Building background knowledge for academic achievement*. Alexandria, VA: Association for Supervision and Curriculum Development.
- Marzano, R. J., Pickering, D. J., & Pollock, J. E. (2001). *Classroom instruction that works*. Upper Saddle River, NJ: Pearson.
- Moje, E., B. (2006). Integrating literacy into the secondary school content areas: An enduring problem in enduring institutions [Electronic Version]. [www.umich.edu/~govrel/adoles\\_lit/moje.pdf](http://www.umich.edu/~govrel/adoles_lit/moje.pdf)-. Retrieved August 20, 2008.
- Monroe, E. E. (1997). Effects of mathematical vocabulary instruction on fourth grade students. *Reading Improvement*, 34, 120-132.
- Murray, M. (2004). *Teaching mathematics vocabulary in context*. Portsmouth, NH: Heinemann.
- National Writing Project, & Nagin, C. (2003). *Because writing matters: Improving student writing in our schools*. San Francisco: Jossey-Bass.
- NCTM. (2000). *Principals and standards for school mathematics*. Reston, VA: Author.
- Pajares, M. F. (1992). Teachers beliefs and educational research: Cleaning up a messy construct. *Review of educational research*, 62, 307-332.
- Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.
- Roe, B. D., Stoodt, B. D., & Burns, P. C. (2001). *Secondary school literacy instruction: The content areas*. Boston: Houghton Mifflin Company.
- Rubenstein, R. N. (2007). Focused strategies for middle-grades mathematics vocabulary development. *Mathematics Teaching in the Middle School*, 13(4), 200-207.
- Rubenstein, R. N., & Thompson, D. R. (2002). Understanding and supporting children's mathematical vocabulary development. *Teaching Children Mathematics*, 9, 107-112.
- Schon, D. A. (1995). *Frame reflection: Toward the resolution of intractable policy controversies*. New York: Basic Books.



- Steele, D. F. (2005). Using writing to access students' schemata knowledge for algebraic thinking. *School Science and Mathematics*, 105(3), 142-154.
- Steele, D. F. (2007). Understanding students' problem-solving knowledge through their writing. *Mathematics Teaching in the Middle School*, 13(2), 102-109.
- Tall, D. O. (1988). Concept image and concept definition. In J. d. Lange & M. Doorman (Eds.), *Senior secondary mathematics education* (pp. 37-41.). Utrecht: OW&OC.
- Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.
- Thompson, D. R., & Rubenstein, R. N. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. *Mathematics Teacher*, 93(7), 568-574.
- Usiskin, Z. (1996). Mathematics as a language. In P. Elliott, C. & M. Kenney, J. (Eds.), *Communication in mathematics, K-12 and beyond, NCTM 1996 Yearbook* (pp. 231-243). Reston, VA: National Council for Teachers of Mathematics.
- Vacaretu, A. S. P. (2008). Reading texts & writing problems to improve problem solving. *Mathematics Teacher*, 101(6), 451-455.
- Vygotsky, L. S. (1962). *Thought and language*. Cambridge, MA: The MIT Press.
- Zinsser, W. (1988). *Writing to learn*. New York: Harper & Row.

## Pre-service Teachers in Mathematics Lesson Study

Elizabeth A. Burroughs<sup>1</sup> & Jennifer L. Luebeck  
Montana State University

**Abstract:** *This paper presents qualitative evidence to answer the questions, “What are the outcomes of engaging pre-service and in-service teachers in a collaborative lesson study experience” and “How can the outcomes of this experience inform future ways to include pre-service teachers in lesson study?” The data gathered demonstrate that including pre-service teachers in lesson study can introduce them to lesson-building as a process and cross-grades teacher collaboration. It can give them opportunities to be critical thinkers in the context of mathematics education and encourages them to think as teachers. One weakness the pre-service teachers demonstrated was an incomplete understanding of the appropriate use of technology in algebra. Consideration of prior knowledge and anticipation of student responses was lacking among both pre-service and in-service teachers. Overall, the data show that pre-service teachers can contribute to the lesson study process as researchers.*

**Keywords:** jugyokenkyuu; Japanese lesson study; pre-service teachers; reflective thinking; technology

### Introduction

Teachers in Japanese schools have attributed much of their professional growth to the practice of *jugyokenkyuu*, translated as *lesson study* (Murata & Takahashi, 2002; Perry & Lewis, 2003; Stigler & Hiebert, 1999). This professional development model is used systematically to deepen content knowledge, increase understanding of pedagogy, and develop one’s ability to observe and understand student learning (Murata & Takahashi, 2002; Perry & Lewis, 2003; Stigler & Hiebert, 1999). Lesson study is a process for creating deep and grounded reflection about the complex activities of teaching that can be shared and discussed with other members of the profession (Fernandez & Chokshi, 2002). In lesson study, teachers spend about 20 hours working collaboratively, in teams of four to seven, to develop an over-arching lesson theme, plan, teach, observe, critique, and revise a lesson that supports this theme (Perry & Lewis, 2003). Research on the effects of American lesson study on teacher practice and student achievement is still in the nascent stage. Few publications provide descriptions of how to include pre-service teachers in lesson study (Fernandez, 2002; Hiebert, Morris, & Glass, 2003, Post &

---

<sup>1</sup> Department of Mathematical Sciences  
Montana State University  
PO Box 172400  
Bozeman, MT 59717-2400  
burrough@math.montana.edu

Varoz, 2008) and a review of literature found no examples of research results on the effects of lesson study on pre-service mathematics teacher education. The present study explores the effectiveness of engaging pre-service teachers in mathematics lesson study and suggests both benefits and challenges for pre-service teachers who engage with practicing teachers in lesson study. The results of this research demonstrate that pre-service teachers can participate in and contribute to lesson study in meaningful ways. The findings also reveal the limits of pre-service teachers' knowledge of teaching and suggest ways to improve future collaborations between pre-service and in-service teachers.

One impetus for including pre-service teachers in lesson study was to enable them to gain rich and authentic research experiences in mathematics teaching and learning. At the university where we teach, graduation requirements mandate that all students conduct independent research as part of their program of study. A few upper-division research courses are available in mathematics, as are others in other content disciplines, but most such courses are neither accessible nor relevant to students in the teacher preparation program. To address this need, we significantly revised our middle school mathematics methods course to include research experiences for pre-service teachers, beginning with a series of targeted classroom observations and task-based student interviews. The final research experience was completion of a lesson study cycle involving both pre-service and in-service teachers.

Why would it be important to include pre-service teachers in lesson study? In 1999, Stigler & Hiebert noted that "lesson study is a new concept for teachers entering the profession. If undergraduate methods courses were restructured to introduce students to collaboratively planning and testing lessons, new teachers would be ready to assume leadership roles more quickly" (p. 158). Fernandez (2002) reports that pre-service teachers in Japan frequently conduct lesson study as part of their student teaching: "They will prepare a study lesson in collaboration with their university-based mentors and the teacher with whom they have been assigned to work at their school site. They will then teach the lesson in this school, and all the teachers in the building, the university mentors, and other student teachers will come observe" (p. 395).

## **Research Questions**

To assess the effectiveness of engaging pre-service teachers in lesson study, we posed the following research questions:

1. What are the outcomes of engaging pre-service and in-service teachers in a collaborative lesson study experience?
  - a. What do pre-service teachers perceive as beneficial about the experience?
  - b. What do pre-service teachers view as challenging about the experience?
  - c. What other outcomes can be identified?
2. How do the outcomes of this experience inform future modifications to the lesson study process that includes both pre-service and in-service teachers?

Based on our experience with pre-service teachers in methods courses and related fieldwork, we expected that pre-service teachers would be able to engage professionally with in-service teachers. We expected their strength would be in their mathematical understanding, while their limited classroom experience would make it difficult for them to anticipate student responses and misconceptions or to estimate appropriate timing of lesson elements. This expectation was affirmed by experts in the field (J. Hiebert, personal communication, October 27, 2008) who predicted that anticipating student responses, a crucial part of lesson study, would be difficult for pre-service teachers.

## **Method**

This research followed a qualitative design using a phenomenological perspective as we sought to understand the experience of lesson study as perceived by pre-service teachers. Evidence was gathered from one section of a middle school mathematics methods course offered in Spring 2009 at the university where we teach. The course is required for all pre-service secondary teachers and for pre-service elementary teachers earning a 15-credit concentration in mathematics. The course has been offered for nearly 20 years, but not typically in conjunction with a field experience and never before in a research format. We continued to revise the course as it played out, following Hiebert's (2003) model for including pre-service teachers as transparent participants in such a process:

A second strategy that can help prospective teachers treat lessons as experiments is based on the fact that the knowledge, dispositions, and competencies that enable prospective teachers to treat lessons as experiments parallel, quite closely, the knowledge, dispositions, and competencies that instructors must develop collaboratively as the courses themselves are improved. The process of course improvement in which the instructors are engaged can be made transparent for prospective teachers so that they can see how the courses they are taking are being planned, evaluated, and revised. This provides an image of how the process can work to generate knowledge for, and improve, teaching. (p. 216)

All 24 pre-service teachers enrolled in the course (eight secondary and sixteen elementary) were considered subjects in this research, representing a sample of convenience. Throughout their weeks of involvement in lesson study, pre-service teachers were given assignments as part of their course requirements in which they were asked to critique and reflect on their research activities and the lesson study experience. Elements of their participation in the lesson study cycle were video-recorded as well. All data reported are from these written artifacts and video-recorded sessions.

The data on which we base this analysis come from four distinct sources:

1. Lesson study artifacts contributed by pre-service teachers as part of their assignments.
2. Written responses to open-ended questions given to pre-service teachers at various points in the course.
3. Transcribed statements made by pre-service teachers during video-recorded lesson study sessions.
4. Oral and written comments from the semester-end workshop attended by the pre-service and in-service teachers.

Mirroring the development of the course itself, we adjusted our questions for pre-service teachers as the lesson study process unfolded and themes of interest emerged. To the extent possible, we followed Glaser's (1978) method of constant comparison, using initial data to inform the development of questions and the collection of new data. Written responses and classroom assignments were analyzed to develop initial codes; these were eventually refined and grouped into categories through a cyclic process involving both researchers. Validity was further addressed through the triangulation of multiple data sources (assignments, open responses, and

interviews) collected over time. Pre-service teachers were aware that we were using their responses to analyze and assess the lesson study process as well as their own performance.

We were integrally involved in this lesson study experience as researchers, coordinators, and participants. We directed the project that supported the in-service teachers' exploration of lesson study. One researcher was also the instructor of the methods course. Both were acquainted with many of the pre-service teachers from past coursework and with the in-service teachers from prior projects. This provided a level of familiarity and trust that supported the authenticity of the data gathered about the phenomenon of collaboration in lesson study.

### **The Lesson Study Experience**

In lesson study, teachers spend about 20 hours working collaboratively in teams of four to seven to complete a full cycle. In separate sessions, they develop an over-arching lesson theme, then research, plan, teach, observe, critique, and ultimately revise and re-teach a lesson presented to students (Perry & Lewis, 2003). The group begins by developing a broad goal for students and identifying a lesson that will help them reach that goal. After researching all aspects of the lesson's content and different ways to approach it, the teachers collaboratively plan the lesson with careful attention to teacher actions and to potential student misconceptions and responses. One group member teaches the lesson while the others observe and gather data about the lesson and its effectiveness. Finally, the group critiques the experience, revises the lesson based on their data, and may re-teach it. For a more detailed description of conducting lesson study, consult Lewis (2002).

The lesson study cycle at the focus of this research was conducted by a core of five middle school mathematics teachers from the two middle schools serving the local district. One sixth and one eighth grade teacher from the first school joined one seventh and two eighth grade teachers from the second school. Both schools use the same curriculum. All five teachers were experienced educators with a minimum of 14 years in the classroom. However, they were relatively new to lesson study, having completed one cycle the previous fall as part of a professional development project. The 24 pre-service teachers were new to lesson study and had only a limited relationship with this group of teachers and their students. Each had completed several hours of observation in two or three of the teachers' classrooms prior to beginning the lesson study cycle.

We sought to create a realistic lesson study experience for pre-service teachers without compromising completion of the full lesson study cycle for the in-service teachers. The model was confined by the logistical constraints of accommodating both pre-service and in-service teachers' schedules. Teachers' time is a limiting factor in conducting lesson study in the U.S. (Campbell, 2003; Abel & Sewell, 1999); adding pre-service teachers' college schedules into the mix multiplied the complications. Our eventual solution was to commit the pre-service teachers and in-service teachers to three late afternoon face-to-face meetings and to supplement the spaces between with written documents and digitally recorded discussions.

The entire group of pre-service teachers participated in the first stage of the lesson study cycle by attending an after-school session where the in-service team reviewed their broad goal and chose a content topic for the current cycle. They had focused on slope in the previous cycle, and chose to return to this topic in the context of parallel and perpendicular lines, even though it was not typically taught within the current unit. The pre-service teachers observed this meeting, sitting in a perimeter around the five in-service teachers who were seated at a table in the center.

This design respected the wishes of the participating in-service teachers, who wanted to continue their own lesson study experience as an intact team, rather than separating to lead smaller groups of pre-service teachers. While this placed limitations on the pre-service teachers' lesson study experience, it provided an authentic view of teachers engaging professional activity.

The pre-service teachers were assigned to carry out the research stage of the lesson study cycle. The methods course instructor assigned pairs of pre-service teachers to investigate specific aspects of mathematical content, prerequisite knowledge, and potential misconceptions related to the lesson topic. Each pre-service team developed a one-page summary of their research and gave oral presentations to the core group of five teachers in a second after-school meeting. The following week, the teacher team used this information to refine the content and focus of the research lesson as pre-service teachers observed in a third face-to-face session.

The actual development of the lesson took place without the pre-service teachers. This was a logistical choice due to conflicting schedules among the five in-service teachers and the 24 pre-service teachers. The teacher team completed a four-column planning document (Lewis, 2002) which allowed them to make a record of specific lesson tasks, procedures, and questions along with anticipated student responses and teacher notes. In the next methods class, the pre-service teachers were provided with this lesson plan document which allowed them the opportunity to make observations and suggestions. We shared these comments with the teacher team during their final lesson planning session.

Because of limitations of space and time, only two pre-service teachers observed the research lesson and participated in the teacher team's subsequent debriefing session. The impressions they shared with their classmates served to enhance the experience of the remaining pre-service teachers, who watched these sessions via video during their methods class later that day. After watching the video, the pre-service teachers had an opportunity to critique both the lesson and the debriefing session where the teacher team analyzed and revised the lesson. A few days later, the pre-service and in-service teachers reconvened at a Saturday workshop where they viewed and discussed video segments of a re-teaching of the lesson and debriefed the entire lesson study experience.

## **Interpretation of Data**

We collected qualitative data from written assignments linked to the lesson study cycle, open-ended reflections about the lesson study process, transcripts from video-recorded lesson study sessions, and end-of-course focus group interviews with both pre-service and in-service teachers. Many of the pre-service teacher responses arose out of group discussion, while others represented individual reflection. The following discussion illuminates the most significant themes that emerged from this compiled data. The results and interpretations drawn from the data may not be representative of methods courses or pre-service teachers in generalized populations. However, the findings demonstrate that pre-service teachers can both contribute to and gain knowledge from participation in the lesson study process.

### *What Pre-service Teachers Learned about Lessons*

The pre-service teachers made specific observations about various elements of the research lesson. In this respect, participation in lesson study richly supported the traditional

methods course goal of examining important elements of lesson planning and implementation. Watching the planning and teaching of a lesson also concretely demonstrated the importance of planning when teaching. Several components of the planning process are discussed below.

**Lesson-building as a process.** The pre-service teachers acquired new ideas about how lessons are developed and enacted: “Seeing lesson study and the development of the lesson was one of the most valuable aspects of the course.” They noted “all the steps involved in preparing a good lesson” and “the amount of work that actually goes into lessons.” Approaching a lesson as a process building up to a focal idea was new to them. They recognized that an important feature of a successful lesson is to “focus on the one thing you want them [students] to do in the lesson.” They also recognized the importance of placing lessons in context:

Teachers had a great idea with the slope, but I think that either they should have used the lesson study when they were learning slope or changed the subject to what the students were currently working on.

**Key issues in planning lessons.** The four-column lesson template was an effective tool for encouraging pre-service teachers to view a lesson as a carefully sequenced integration of content and pedagogy and for engaging them in serious inquiry about both planning and implementing lessons. They identified issues of timing and sequencing. For example, they noted that the lesson was too full for the allotted 50 minutes, wondering “Is there enough time?” and “What will happen if they don’t achieve the necessary discoveries in the [warm-up]....How do you move to Challenge #1?” Some pre-service teachers suggesting that time was not well-spent in the lesson planning sessions and should have focused more directly on fine-tuning the lesson:

Parts of the actual lesson at times were not thought through. An example of this is the lines they chose to have the students graph. The four lines the students put into their calculators were so close together that it was hard to tell them apart. This was a sign that they may have needed more time to put this lesson together.

**Assessing prior knowledge.** Pre-service teachers also recognized the need to attend to student understanding throughout the lesson. They wondered if the lesson was introduced effectively, because it assumed and relied on students’ prior knowledge about parallel and perpendicular lines. Pre-service teachers worried, “Do students know perpendicular?” and wondered if students could “identify perpendicularity just by looking at the calculator screens versus observing simply that they intersect.” They recognized potential pitfalls: “Do students get that if they make two [perpendicular] lines to one original line that the two [perpendicular] lines are parallel to each other?” and referenced broad goals: “Do students have a deeper understanding of slope now?” These comments seem particularly astute in hindsight, because one conclusion the teacher team reached following the lesson study cycle was that students didn’t really understand the difference between two lines that are perpendicular and two lines that intersect.

**Anticipating student responses.** We had predicted that pre-service teachers would have difficulty anticipating student responses to the research lesson. In fact they anticipated student responses that did not play out in the classroom, and failed to anticipate responses that emerged during the teaching of the lesson. This deficit could be the result of a general lack of knowledge of student thinking or limited experience with the particular students involved in this research lesson. It is worth noting that the in-service teachers also neglected the issue of anticipating student responses, a gap in planning that the pre-service teachers were later able to identify:

The aspect of this process that appeared very difficult for these teachers was focusing on what gaps of knowledge or misconceptions their students really might

have about slope...instead these teachers continued to focus on what was coming next in the curriculum.

A pre-service teacher noted as a weakness “the teachers’ abilities to predict students’ successes and difficulties. Even though students’ struggles were discussed previously, strategies were not discussed to prepare for such struggles.” Another saw a lack of formative feedback, concluding that “there was no data as to how many students succeeded in either lesson presentation.”

**Appropriate use of technology.** Pre-service teachers showed an apparent lack of understanding regarding the instructional purpose of technology in the lesson. One pre-service teacher recognized that “If you have the kids hand graph, messier kids won’t see that [the lines] are parallel or perpendicular” and added “But does graphing with the calculator hinder understanding?” Even after identifying a specific need for the calculator, the pre-service teacher remained suspicious of technology use in general. Another pre-service teacher suggested that students “do this lesson manually with graph paper to make sure they acknowledge everything and understand it, then do a problem where they can use a calculator.” A third wondered, “Does using the calculators help students learn? It seems they may cause more concerns, especially when used to introduce the concept for the first time.” Such reactions hint at a preconceived notion among the pre-service teachers that technology can not be used to build mathematical understanding among students. It may be true that in this lesson technology was not used effectively as a tool to build mathematical understanding, but the pre-service teachers did not offer comments about its *appropriate* use; they were suspicious of *any* use of technology. This indicates a need to address technology use more thoroughly in the methods course.

#### *What Pre-service Teachers Learned from Lesson Study*

**Contributing researchers.** During the research phase of the lesson study cycle, the pre-service teachers investigated how slope and parallel lines are addressed in research journals, practitioner articles, and other curricula, and orally presented their findings to the teacher team. They were surprised at the enthusiastic reception they received. One student said, “I had presumptuously assumed these teachers knew and had considered [already] all we had to offer.”

**Cross-grades collaborators.** The pre-service teachers valued the opportunity to actively interact with practicing teachers. They also recognized the “importance of opening up dialog between grades.” They saw the exchange of ideas between 6<sup>th</sup>-, 7<sup>th</sup>-, and 8<sup>th</sup>-grade teachers as parallel to their own experience in a methods course combining elementary and secondary teachers. One student noted:

Collaboration between teachers across different grade levels is very important for both teachers and those of us who are observing. It gives each teacher for different grade levels insight into what their students need to know and it is important for us to realize that as well.

**Critical colleagues.** Pre-service teachers analyzed the lesson study cycle with insight and often provided sound advice for improvement. One of their primary concerns was the selection of the person who would teach the research lesson. For the sake of expediency, the teacher team had pre-determined who would teach and re-teach the lesson. The pre-service teachers saw that assigning a teacher to the lesson too early caused the others not to take group ownership of the lesson. One student advised, “I think that not knowing who is going to be teaching the lesson is



ideal; this way all the teachers can research and go through the process as though they will be the ones teaching the lesson.” Another observed that assigning a teacher immediately prior to the lesson “would have added another level to the lesson, making it more effective.”

The pre-service teachers were also able to assess the social dynamics behind the in-service teachers’ collaboration, as with the comment, “I think that the teachers did a pretty good job of getting together and talking about the subject to be taught. I do not think they worked as a team very well....Here in [this team] I never heard the teachers say ‘we’ and ‘our lesson.’”

**Reflective practitioners.** The pre-service teachers seemed to take to heart their role as co-developers of the new research aspects of the methods course. They offered many comments demonstrating their ability to reflect not just on the results of this particular lesson study, but also on participating in lesson study as a research experience. One pre-service teacher concluded: “I really enjoyed the research aspect of this class, as well as working with the teachers. I gained a lot of insight from the observations and lesson study process. I would have enjoyed incorporating more if it into the class.” They were clearly conscious of their role as observers rather than participants during the lesson study cycle. “Looking back on the experience I wish that [pre-service teachers] had had a little more input into the lesson. Although we did have some input I felt that we were largely observers of the process, with the decisions made over our heads.”

Over the course of the lesson study, the pre-service teachers became more invested in the process and proved themselves quite capable as a group of understanding and valuing the lesson study process. They were able to see that lesson study, even though imperfect, was worthwhile for the teacher team as they “gained insight on how the lesson directly influences the students’ learning.” However, they were also cognizant of the need to embed the practice in a broader context:

As the teachers planned the lesson, I did believe that we slightly lost sight of our overarching goal and became too focused on the topic alone. I thought we needed to stop and consider how the activities were promoting ‘confident, independent problem solvers’ as well as how the goal of the lesson study process was being achieved.

A final student comment reflects on the pitfalls of doing lesson study for its own sake, a piece of wisdom that can be applied to any effort at teacher professional growth and development:

Another important thing to note is that lesson study by itself does not improve teaching. If the group of teachers does not put the time and effort into it, or do not have the knowledge or don’t research their topic, the study is ineffective.

### **Implications for the Preparation of Mathematics Teachers**

The above discussion illuminates the nature of pre-service teachers’ contributions to a mathematics lesson study cycle. In summary, our findings suggest that pre-service teachers can indeed participate meaningfully with in-service teachers in the practice of lesson study. They engaged in professional dialog as peers and perceived themselves as valued contributors. At the same time, they learned valuable lessons about teaching, learning, and collaboration in a mathematics community.

Reflective thinking emerged as a strength among the pre-service teachers as they critiqued both the research lesson and the lesson study process. At the lesson level, the lesson study cycle pushed the methods course experience beyond merely providing ideas about

planning and implementing future lessons to developing a deep, experiential understanding of lesson development. In the words of one observer, it helped make a somewhat covert planning process highly visible to prospective teachers. At a process level, the pre-service teachers gained enough knowledge about what lesson study should be to recognize the shortfalls of the particular lesson study in which they participated. This came partly through readings and watching video of another lesson study conducted by teachers in a different district. The pre-service teachers were also able to critique their own involvement in the course model chosen by the course instructor to implement the lesson study.

As anticipated, the pre-service teachers are not yet equipped to accurately predict student responses and address misconceptions. However, the in-service teachers were also weak in this area, and the pre-service teachers recognized that as they observed the lesson study process. Overall, this should not be considered as a limiting factor in the use of this model for including pre-service teachers in lesson study.

A more significant limitation was the fact that logistical obstacles effectively removed the pre-service teachers from the detailed process of planning the research lesson, one of the most critical parts of the process and a rich example of applying the mathematical and pedagogical knowledge needed for teaching. Even in sessions they could attend, watching and not contributing to decisions left several pre-service teachers feeling powerless. They participated according to our expectations, but were left feeling less like researchers or participants than simple observers. This is contrary to the philosophy that guided our revision of this course.

In the future, we will strengthen the role of pre-service teachers in planning the actual components of the research lesson. There are now two lesson study teams prepared to work with pre-service teachers, allowing a better balance of pre- and in-service teachers on a team. Teachers have also offered to come to the university campus to conduct their lesson study sessions during the methods course, which will help to reduce logistical problems. Other models may also be explored, such as creating smaller mixed groups of pre- and in-service teachers for multiple lesson studies.

This examination of pre-service teacher involvement in lesson study has clarified the true capabilities of pre-service teachers as reflective and collaborative pre-professionals. It has also revealed the high levels of appreciation and acceptance extended by in-service teachers to their colleagues-in-training. The practice of lesson study in our methods course will continue and, we expect, will prompt further improvements in the classroom research experiences we provide for our pre-service teachers.

## References

- Abel, M., & Sewell, J. (1999). Stress and burnout in rural and urban secondary school teachers [Electronic version]. *Journal of Educational Research*, 92(5), 287–293.
- Campbell, C. (2003). *Translating Japanese lesson study in United States high schools*. Unpublished doctoral dissertation, University of California, Los Angeles.
- Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education* 53, 393–405.

- Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform-minded teaching? *Mathematical Thinking And Learning*, 7(4), 265–289.
- Fernandez, C., & Chokshi, S. (2002). A practical guide to translating lesson study for a U. S. setting. *Phi Delta Kappan*, 84(2), 128–134.
- Glaser, B. & Strauss, A. (1967). *Discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing Company.
- Hiebert, J., Morris, A.K. & Glass, B. (2003). Learning to learn to teach: An “experiment” model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education* 6: 201–222.
- Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia, PA: Research for Better Schools, Inc.
- Lewis, C., Perry, R., & Herd, J. (2004). A deeper look at lesson study [Electronic version]. *Educational Leadership*, 61(5), 18–22.
- Murata, A., & Takahashi, A. (2002). Vehicle to connect theory, research, and practice: How teacher thinking changes in district-level lesson study in Japan. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Noony (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Athens, Georgia. (ERIC Document Reproduction Service No. ED471780)
- Perry, R., & Lewis, C. (2003, April). *Teacher-initiated lesson study in a Northern California district*. Paper presented at the Annual Meeting of the American Educational Research Association, Chicago, Illinois. (ERIC Document Reproduction Service No. ED478391)
- Post, G. & Varoz, S. 2008. Supporting teacher learning: Lesson-study groups with prospective and practicing teachers. *Teaching Children Mathematics* 14(8): 472–478.
- Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world’s teachers for improving education in the classroom*. New York, NY: The Free Press.

## Internal and External Comments on Course Evaluations and Their Relationship to Course Grades

H. Smith Risser<sup>1</sup>

Montana Tech of the University of Montana

**Abstract:** *The validity of student evaluations of courses and the relationship between evaluations and course outcomes has frequently been examined. Since many course evaluations give students an opportunity to provide answers to open-ended questions in addition to giving Likert scale ratings, it is important to understand the relationship between these responses and course outcomes. This study examined the relationship between student responses to open ended questions (specifically whether they attributed their achievement to factors within their control or factors not within their control) and their outcomes in the course. The results of the study indicate that students that identified external factors (e.g. the professor) as responsible for their success in the course had significantly higher course outcomes than students that identified external factors as responsible for their failure in the course. No differences were detected in course outcomes for students that made external comments on their course evaluations versus those that made internal comments.*

**Keywords:** course evaluations; course grades; evaluation; Likert scale; open ended questions; student achievement; student failure; student failure;

### Introduction

A person's locus of control refers to the beliefs they hold about the relationship between their actions and outcomes (Lefcourt, 1991). Locus of control can be internal or external to the person in question. An individual with an internal locus of control will typically perceive their successes and failures as a result of their own actions, while an individual with an external locus of control will typically ascribe their successes and failures to factors like luck or a powerful other (Lefcourt, 1991). Research has indicated that an individual's beliefs about the locus of control can vary based upon the context examined

---

<sup>1</sup> 1300 W. Park  
Butte, MT 59701  
Phone: 406-496-4581  
hrisser@mtech.edu

and that the locus of control in one context may not relate to attitudes and/or behaviors in a different context. Ang and Chang (1999) looked at an individual's perception of locus of control for social outcomes and outcomes on particular tasks and the relationship of the locus of control in both areas to their individual goals. The results of the study indicated that participants' internal and external locus of control for particular tasks were predictive of their achievement goals, but not their social goals. Similarly, participants' internal and external locus of control for social outcomes were predictive of their social goals, but not their achievement goals. This suggests that an individual's locus of control in a specific area is related to their goals and/or behaviors in that area, but not in other areas.

Thus, locus of control scales specific to a variety of contexts such as health contexts (Hill & Bale, 1980; Wallston & Wallston, 1981), professional contexts (Spector, 1988), and academic contexts (Crandall, Katkovsky, & Crandall, 1965) have been developed. An individual's beliefs about the causes of outcomes in their lives are often neither entirely internal nor entirely external, so locus of control scales typically measure an individual's locus of control on a continuum. One end of the continuum represents an individual with a wholly internal locus of control and the other end an individual with a wholly external locus of control. Previous research has examined the relationship between internal locus of control and positive health behaviors (Graffeo & Silvestri, 2006), lower levels of depression (Njus & Brockway, 1999), and increased self-esteem (Chubb, Fertman, & Ross, 1997).

In academic situations, locus of control is related to student perceptions and beliefs. In an examination of approximately a hundred pre-service teachers, Akinsola (2008) discovered that participants' locus of control was related to their perceptions of their competence and attitudes towards problem solving. Participants in the study with an internal locus of control perceived that they were more competent in problem solving than students with an external locus of control. An individual's beliefs about the locus of control in academic situations can also affect their perceptions of the roles of student and professor in the classroom. In a case study of four college students, Wang (2005) found that two students with a highly internal locus of control felt personally responsible for their learning in the course. In their comments, they blamed their own behaviors or beliefs for their difficulties in the course. In contrast, the students with a highly external locus of control felt the instructor was primarily responsible for making sure that learning would occur in the course. Both of the external students blamed the instructor when they did not understand aspects of the course or the material.

Internal locus of control has also been shown to be related to positive academic behaviors like increased effort (Carbonaro, 2005) and better use of particular learning strategies (Kesici, Sahin, & Aktruk, 2009). Students with an external locus of control believe that course outcomes will be affected most by external factors, thus there is little reward for extra effort. Students with an internal locus of control, however, believe that course outcomes can be changed by hard work or seeking out help and will therefore be more likely to put in additional effort (Hunter & Barker, 1987).

Based on the relationship between locus of control and academic behaviors it is not surprising that locus of control is also related to academic outcomes. Flouri (2006) found that a more internal locus of control at age 10 was related to higher educational attainment at age 26. Nordstrom and Segrist (2009) found that students with a more internal locus of control were more likely to intend to enroll in graduate school than students with an external locus of control. In a study of approximately 3,000 freshmen college students, Gifford, Briceño-Perriott, and Mianzo (2006) found that locus of control was a significant predictor of cumulative college GPA. Freshman that were classified as having a more internal locus of control had a significantly higher GPA than those students classified as having a more external locus of control. Similar findings have resulted from studies by Wood, Saylor, and Cohen (2009) and Agnew, Slate, Jones, and Agnew (1993).

The purpose of this study was to examine the relationship between what factors (internal versus external) students identify as contributing to their achievement in an introductory mathematics course and their course outcomes.

Student ratings on evaluations have been shown to be related to a variety of course and instructor variables. For instance, instructors of science courses receive lower course evaluations than instructors of humanities courses (Marsh & Roche, 1997) and instructors with small enrollments tend to receive better evaluations than instructors of classes with larger enrollments (Liaw & Goh, 2004). Based on these results, it would be expected that student variables like locus of control would influence course ratings as well.

A study by Grimes, Millea, and Woodruff (2004) examined the relationship between course evaluations and locus of control, in a macroeconomics course. Students' locus of control was measured using Rotter's I-E scale. Then each student completed an evaluation of the course. Students were asked to evaluate their learning, the course, and the instructor with responses to Likert scale questions. In this study, students with a highly internal locus of control were more likely to give the instructor a positive rating than students with an external locus of control.

Ramanaiah and Adams (1981) examined the relationship between expected grade in the course and course evaluation ratings for approximately two hundred undergraduate students in an introductory psychology course. Rotter's I-E scale was administered to the students at the midpoint of the semester in order to classify the students as either having an internal or an external locus of control. Although Rotter's I-E scale produces a continuous score, Ramanaiah and Adams used the group median score to divide the participants into two groups; one with an internal locus of control and the other with an external locus of control. At the end of the semester, student evaluations of the course and instructor and their expected grade in the course were collected. Then, correlations between expected grade in course and ratings of the instructor and the course were computed for both groups of students. The results of the study indicate that there were no significant differences in the relationships between course evaluations and expected course grades for students with an internal locus of control and those with an external locus of control. In other words, the results of this study indicate that the relationship

between expected grades and how a student evaluates a course does not depend on the students' locus of control.

There are two main difficulties with this study. First, in order to make comparisons between groups, Ramanaiah and Adams classified students as either internal or external based on their score on the locus of control continuum. Since individuals have neither completely internal nor completely external locus of control beliefs, classifying participants in this way may have influenced the results. Second, while locus of control is typically thought of as a stable personality characteristic, previous research has indicated that an individual's locus of control can be changed by events (Legerski, Conwall, & O'Neil, 2006) or interventions (Roberts, Zachorchemny, & Cohen, 1992). Thus, an academic event (e.g. failing a test that the student studied very hard for) could change an individual's perception of locus of control. So, beliefs about the locus of control measured in the middle of the semester might not necessarily reflect beliefs about locus of control at the end of the semester.

## **Methods**

### **Sample**

The study was conducted at a 4-year public university in the South. The student body is around 11,000 students primarily women (90%) and has a large percentage of minority students (38%). The university offers four introductory mathematics courses (a quantitative literacy course, two college algebra courses, and a statistics course). These courses are typically taken by non-majors seeking to satisfy the university's general education requirement for mathematics. Each course has a departmental syllabus with a common grading policy and a common final exam. This made it possible to use course grade and the grade on the final exam as outcome variables despite the fact that the participants did not all have the same instructor.

Approximately eight hundred students were enrolled in the seventeen sections (3 quantitative literacy, 2 lower level college algebra, 4 upper level college algebra, & 8 statistics) of the courses offered in the spring semester of 2006. Approximately four hundred students consented to participate in the study, however only two hundred students completed the entire second measure. The participants that did complete the entire second measure ranged in age from 18 to 56 with an average age of 21.48. Approximately 96% ( $N = 192$ ) of the participants were female and approximately 38% of the participants were minorities ( $N = 75$ ).

### **Recruitment and Study Design**

Students were recruited through visits to their mathematics classes during the first two weeks of the semester. Participants in the study were asked to complete two measures and to give the researcher permission to obtain their academic record and course

outcomes. The first measure concerning the participants' background was administered during the recruitment visit. The second measure concerning participants' experiences in the course was administered during a visit in the final two weeks of the semester. The measures included both quantitative and qualitative data. At the end of the semester students' grades on the departmental final exam and their grade in the course were also collected. No compensation was given to students that chose to participate in the study.

### **Coding Locus of Control**

In the second measure participants were asked to answer the following open-ended question, "What would have helped you to be more successful in this course?". The responses to this question were analyzed in order to determine whether the factors the participant listed were internal or external to their locus of control.

In order to classify the comments as either internal or external, a coding scheme based on the Intellectual Achievement Responsibility (IAR) measure (Crandall, Katkovsky, & Crandall, 1965) was developed. Academic behaviors that indicated either an internal or external locus of control were identified from the questions in the IAR. Some of the internal behaviors included in the IAR are studying, paying attention, being careful, making effort, and inherent skill. Some of the external factors given as choices in the IAR are instructions, luck in the form of an easy test or question, or the behavior of an external person like a friend or teacher.

The behaviors identified from the IAR were then used to code the participant answers. Since the question was open-ended, it was possible for a response to indicate both an internal and external locus of control. For example, the response, "I feel confident in this math course and I feel like my teacher helped me to be very successful. I could have studied [sic] a little more." was coded as both an internal and external.

In addition to classifying student comments as internal and/or external, comments were also classified as either positive or negative. A positive comment would be one in which a student attributed their success in the course to a particular factor (either internal or external). A negative comment would be one in which a student attributed their failure to a particular factor. This additional classification was included because research has shown that students may also perceive one set of factors as responsible for their successes and another set of factors as responsible for their failures Lei (2009).

In total, ninety-four comments were coded as being internal. Seventy-eight were internal negative and eighteen were internal positive. Two comments were coded as both internal negative and internal positive. Ninety-eight of the responses were coded as external. Thirty-five responses were external positive and sixty-three were external negative with one response being coded as both external positive and external negative. When comparisons between groups were done, any comments that were coded as being members of both groups were not included in the analysis to prevent the groups from being dependent.



Some of the responses were also coded as undetermined. For these responses, the coder was unable to determine the intent of the participant from the sample. Only fifteen of the two hundred responses could not be classified. A sample of the responses for each type of code is given in Appendix A. Each response was coded by two separate coders for reliability. Cronbach alphas computing coder reliability ranged from 0.62 to 0.93.

### **Outcome Variables**

Measures of course outcomes were recorded for the participants that received a grade in the course. The course grade for each participant was recorded. Approximately 96% of the participants ( $N = 191$ ) took the final examination and received a grade in the course. The course grades ranged from 0 to 4 with A = 4, B = 3, C = 2, D = 1, and F = 0. The average course grade for the participants was a 2.9. The score on the department final exam was also used as an outcome variable. Participants' scores on the departmental final exams ranged from 17.65% to 100% with an average of 64.6%.

### **Results**

The relationship between demographic information (i.e. age and ethnicity) and the outcome variables (i.e. course grade and final exam grade) were examined. Since no significant relationships were determined, these variables were not included in the regression models. The relationships between gender and the outcome variables were not examined since so few participants were male.

### **Comparisons between groups**

A t-test was computed to look for differences in course outcomes for students that made external comments on their course evaluations and those that made internal comments. No significant differences in either the mean final exam grade ( $M = 65.41$ ,  $M = 63.53$ ,  $t(170) = .67$ ) or the mean course grade ( $M = 3.02$ ,  $M = 2.88$ ,  $t(170) = .85$ ) between the two groups were detected.

A t-test was computed to look for differences in course outcomes for students that made positive external comments and those that made negative external comments. Students that made negative external comments had a significantly lower mean score on the final exam ( $M = 61.80$ ,  $SD = 18.79$ ) than students that made positive external comments ( $M = 72.75$ ,  $SD = 15.32$ );  $t(89) = -2.87$ ,  $p < .01$ . Students that made negative external comments also had a significantly lower mean course grade ( $M = 2.77$ ,  $SD = 1.04$ ) than students that made positive external comments ( $M = 3.47$ ,  $SD = .71$ );  $t(89) = -3.48$ ,  $p < .001$ .

### **Predicting outcomes using comments**

Since it was possible for a student comment to have more than one code associated with it, regression models were computed to determine the relationships between the codes assigned to students' comments and students' grades on both the final exam and the course. Comments that could not be classified as either internal or external were not included in the regression model. There were four possible codes that could be assigned to students' comments; internal positive (IP), internal negative (IN), external positive (EP), external negative (EN). These four codes were used as the independent variables in

the model. Each of the independent variables was assigned either a 0 or 1. For example, a comment that was coded as both internal negative and external positive would have the following values for each of the four independent variables: IP = 0, IN = 1, EP = 1, EN = 0. The dependent variables were the student outcomes in the course.

A regression model was computed with internal positive, internal negative, external positive, and external negative comments predicting the score on the final exam (see Table 1).

$$\text{final exam grade} = 58.76 + 15.51 (\text{IP}) + 2.59 (\text{IN}) + 13.49 (\text{EP}) + 3.15 (\text{EN})$$

The model was significant ( $F(4,190) = 4.88, p < .001$ ) and both internal positive and external positive comments were significant predictors of the final exam grade(see Table 1).

Table 1: Regression predicting final exam grade from student comments

Variable	Parameter Estimate	T obs.	Signif. T
internal positive (IP)	15.51	2.95	0.004
internal negative (IN)	2.59	0.68	0.498
external positive (EP)	13.49	3.21	0.002
external negative (EN)	3.15	0.76	0.447
$r^2 = 0.09$			

A regression model was computed with internal positive, internal negative, external positive, and external negative comments predicting the grade in the course (see Table 2).

$$\text{course grade} = 2.80 + 0.94 (\text{IP}) - 0.13 (\text{IN}) + 0.65 (\text{EP}) - 0.04 (\text{EN})$$

The model was significant ( $F(4,190) = 6.48, p < .0001$ ) and both internal positive and external positive comments were significant predictors of the course grade (see Table 2).

Table 2: Regression predicting course grade from student comments

Variable	Parameter Estimate	T obs.	Signif. T
internal positive (IP)	0.94	3.08	0.002
internal negative (IN)	-0.13	-0.61	0.546
external positive (EP)	0.65	2.65	0.009
external negative (EN)	-0.04	-0.15	0.883
$r^2 = 0.12$			

## Discussion

Based on the results of previous studies (Agnew, Slate, Jones, and Agnew, 1993; Gifford, Briceño-Perriott, & Mianzo, 2006; Wood, Saylor, and Cohen, 2009) we expected that students that credited their success or failure to internal factors would have higher course outcomes than students that explained their achievement using external factors. However, the results of this study did not detect significant differences in mean course outcomes between the two groups. This result may be partially explained by the results of a study by Lei (2009) which found that students were likely to attribute their success to external factors and their failures to internal factors. This analysis did not consider whether the comments were positive or negative, only whether they were external or internal.

Differences in course outcomes were detected between students that made positive external comments and those that made negative external comments. Students that made negative external comments had lower final exam scores and lower course grades than students that made positive external comments. A study by Grimes, Millea, and Woodruff (2004) found that students with a highly internal locus of control were more likely to give the instructor a positive rating than students with an external locus of control. This result indicates that students with an internal locus of control are generally more satisfied with the instruction in a course than students with an external locus of control. Thus a positive comment about the instructor might not be related to an external locus of control. In this study, 22 of the positive external comments (11%) were related to the instructor. A larger sample of students would allow the student comments about the instructor to be separated from other internal and external comments.

In the regression analyses, it was determined that student comments on course evaluations had high predictive values for course outcomes. Further, positive comments (both external and internal) were positively related to both the letter grade in the course and the final exam score. For both models internal positive comments contributed most to predicting the final exam grades and course grades. This implies that students that make internal positive comments will likely have both higher exam grades and higher grades in the course. Since an internal locus of control is usually related to positive academic outcomes this is not a surprising result. It is surprising that negative internal comments are not significant predictors of course outcomes and positive external comments are significant predictors. This suggests that the attribution a student's success in the course to a particular factor is more important than whether the factor is internal or external. It is also interesting to note that negative comments (either internal or external) are negatively related to course grade but positively related to final exam grade.

The primary limitation of this study was that only half of the participants completed the open-ended question on the second measure. A larger sample of student responses would allow the student comments to be further compared by the specific factor (e.g. instructor, course structure) that they attributed their success and/or failure to rather than only classifying the factor as internal or external. In addition, a larger sample might produce a larger number of positive internal comments, allowing for comparisons between the positive internal group and other groups. The low response rate is commonly observed in

open-ended course evaluations (Darby, 2007). Since online course evaluations have been shown to have a higher response rate for open-ended questions than paper evaluations (Laubsch, 2006), one possible resolution to this difficulty would be to administer the open-ended questions online.

The use of teaching evaluations by instructors to improve their teaching and administrators to justify tenure and promotion decisions emphasizes the importance of understanding the relationship between student ratings of a course and course outcomes like student learning and course grades. The results of this study showed evidence of a relationship between the type of comments a student makes on course evaluations and course outcomes. However, the cause of this relationship is not known and merits further exploration.

## References

- Agnew, N., Slate, J., Jones, C., & Agnew, D. (1993). Academic behaviors as a function of academic achievement, locus of control, & motivational orientation. *NACTA Journal*, 37(2), 24-27.
- Akinsola, M. (2008). Relationship of some psychological variables in predicting problem solving ability of in-service mathematics teachers. *The Montana Mathematics Enthusiast*, 5(1), 79-100.
- Ang, R. & Chang, W. (1999). Impact of domain-specific locus of control on need for achievement and affiliation. *The Journal of Social Psychology*, 139(4), 527-529.
- Carbonaro, W. (2005). Tracking students' effort, and academic achievement. *Sociology of Education*, 78(1), 27-49.
- Chubb, N., Fertman, C., & Ross, J. (1997). Adolescent self-esteem and locus of control: A longitudinal study of gender and age differences. *Adolescence*, 32(125), 113-130.
- Crandall, V. C., Katkovsky, W., & Crandall, V. J. (1965). Children's beliefs in their own control of reinforcements in intellectual-academic achievement situations. *Child Development*, 36, 91-109.
- Darby, J. (2007). Open-ended course evaluations: A response rate problem? *Journal of European Industrial Training*, 31(5), 402-412.
- Flouri, E. (2006). Parent interest in children's education, children's self-esteem and locus and control, and later educational attainment: Twenty-six year follow-up of the 1970 British birth cohort. *British Journal of Educational Psychology*, 76, 41-55.
- Graffeo, L. & Silvestri, L. (2006). Relationship between locus of control and health-related variables. *Education*, 126(3), 593-596.
- Gifford, D., Briceño-Perriott, J., & Mianzo, F. (2006). Locus of control: Academic achievement and retention in a sample of university first-year students. *Journal of College Admission*, 191, 18-25.
- Grimes, P., Millea, M., & Woodruff, T. (2004). Grades – who's to blame? Student evaluation of teaching and locus of control. *Journal of Economic Education*, 35(2), 129-148.
- Hill, D. & Bale, R. (1980). Development of the Mental Health Locus of Control and Mental Health Locus of Origin scales. *Journal of Personality Assessment*, 44, 148-156.

- Hunter, M. & Barker, G. (1987). "If at first ...": Attribution theory in the classroom. *Educational Leadership*, 45(2), 50-53
- Kesici, S., Sahin, I., & Aktruk, A. (2009). Analysis of cognitive learning strategies and computer attitudes, according to college students' gender and locus of control. *Computers in Human Behavior*, 25, 529-534.
- Laubsch, P. (2006). Online and in-person evaluations: A literature review and exploratory comparison. *Journal of Online Learning and Teaching*, 2(2), 62-72.
- Lefcourt, H. (1991). Locus of control. In J. P. Robinson, P. R. Shaver, & L. S. Wrightsman (Eds.), *Measures of personality and social psychological attitudes* (pp. 413-500). San Diego, CA: Academic Press.
- Legerski, E., Cornwall, M., & O'Neil, B. (2006). Changing locus of control: Steelworkers adjusting to forced unemployment. *Social Forces*, 84(3), 1521-1537.
- Lei, C. (2009). On the causal attribution of academic achievement in college students. *Asian Social Science*, 5(8), 87-96.
- Liaw, S. & Goh, K. (2004). Evidence and control of biases in student evaluations of teaching. *The International Journal of Educational Management*, 17(1), 37-43.
- Marsh, H. & Roche, L. (1997). Making students' evaluations of teaching effectiveness effective: The critical issues of validity, bias, and utility. *American Psychologist*, 52, 1187-1197.
- Nordstrom, C. & Segrist, D. (2009). Predicting the likelihood of going to graduate school: The importance of locus of control. *College Student Journal*, 43(1), 200-206.
- Njus, D. & Brockway, J. (1999). Perceptions of competence and locus of control for positive and negative outcomes: Predicting depression and adjustment to college. *Personality and Individual Differences*, 26, 531-548.
- Ramanaiah N. & Adams M. (1981). Locus of control and attribution of responsibility for academic performance, *Journal of personality assessment*, 45(3), 309-313.
- Roberts, J., Zachorchemny, C., & Cohen, J. (1992). Change in locus of control following admission in adolescent in-patients. *British Journal of Psychiatry*, 161, 809-815.
- Spector, P. (1988). Development of the work locus of control scale. *Journal of Occupational Psychology*, 61, 335-340.
- Wallston, K., Wallston, B., Devellis, R. (1978). Development of multidimensional health locus of control (MHLC) scales. *Health Education Monographs*, 6(2), 160-170.
- Wang, D. (2005). Students' learning and locus of control in web-supplemented instruction. *Innovative Higher Education*, 30(1), 67-82.
- Wood, A., Saylor, C., & Cohen, J. (2009). Locus of control and academic success among ethnically diverse baccalaureate nursing students. *Nursing Education Research*, 30(5), 290-294.

## Appendix A – Coding examples

### **Internal positive**

“Nothing, I did pretty good.”

“Nothing, I think I made pretty close to a perfect score.”

“I am making an A so I wouldn’t change anything.”

### **Internal negative**

“working fewer hours”

“working additional homework problems to practice for tests”

“If I had read ahead and prepared much more earlier before getting to class”

### **External positive**

“Nothing, I felt absolutely prepared from previous math courses.”

“Nothing. \_\_\_\_ is a great teacher. He/she teaches well and I am successful.”

“The instructor made the class unlike the traditional maths classes, therefore I was able to grasp the techniques easily and work on my own better.”

### **External negative**

“If the instructor explained a little bit more, and if he/she wasn’t so boring.”

“More interactive learning, hands on work”

“Make math more fun and meanful [sic] as in related to real life situation and why we use this kind of math for.”

### **Unclassified**

“not having to take this course”

“Actually get help from a mathematic to improve to a greater level.”



## Problem Posing from the Foundations of Mathematics

Libby Knott<sup>1</sup>

Washington State University

*Abstract: This reflective paper develops a repertoire of questions for teachers to use in their classrooms during episodes of mathematics discussions with and among students. These questions are motivated by an examination of questions posed by Wittgenstein in Zettel, and are connected to underlying tacit assumptions about mathematics, most of which lie subtly below the generally accepted milieu of math-talk. Once classrooms norms have been established to encourage participation by all students in a democratic and just classroom environment, these questions can be used effectively to stimulate meaningful discourse. These questions provide important examples of problem posing designed to encourage student reflection.*

**Keywords:** classroom environment; discourse; history of mathematics; problem posing; mathematical discourse; tacit knowledge; Wittgenstein; *Zettel*

### **Introduction**

Recent developments in mathematics education research have shown that creating active classrooms, posing and solving cognitively challenging problems, promoting reflection, meta-cognition and facilitating broad ranging discussions, enhances students' understanding of mathematics at all levels. The associated discourse is enabled not only by the teacher's expertise in the content area, but also by what the teacher says, what kind of questions the teacher asks, and what kind of responses and participation the teacher expects and negotiates with the students. Teacher expectations are reflected in the social and socio-mathematical norms established in the classroom. But not all teachers are adept at asking the appropriate questions in a way that enhances learning. Often, they have not had experience in classroom settings or as team teachers where this kind of active dialogic discourse has been observable, nor have they had opportunities to practice it themselves. How can one short-cut the process of equipping teachers with the perspectives and skills they need to respond in routine situations when probing and challenging questions are called for? If teachers were to have a repertoire (which some teachers eventually gain through trial and error) of questions and insights, they would perhaps improve the level of cognitive demand and intellectual stimulation in their classrooms for all students, even for those whose classroom participation is infrequent, hesitant or uncomfortable.

To elucidate this kind of repertoire I have gone back into the historical record of mathematical discussions and found, in *Zettel* (Wittgenstein, 1967), a rich source of short phrases or aphorisms which I have matched with probing questions that relate to the foundations of mathematics. This

---

<sup>1</sup> Washington State University  
PO Box 643113  
Pullman, WA 99164-3113  
Email: [lnott@wsu.edu](mailto:lnott@wsu.edu)



resource offers suggestions that correlate with discussions in mathematics classes that I have taught, observed, participated in and analyzed during the last two decades.

### **Background**

Wittgenstein was a major force behind the scenes of some of the most important developments in the philosophy of mathematics (as he was also for the disciplines of Psychology, Philosophy, Logic, Linguistics and Ethics.) (Marion, 1998) During the twentieth century there were so many innovations in science and philosophy that the explanation of the foundations of mathematics by Ludwig Wittgenstein most often goes completely unacknowledged. Wittgenstein matriculated into Cambridge just as Russell & Whitehead's *Principia Mathematica* was printed, and his was one of the first critical minds to study it and see its fundamental weaknesses (Russell & Whitehead, 1956). He was credited with having convinced Bertrand Russell that his joint work was seriously flawed (Ramsey, 1925). Wittgenstein never published a counter explanation, but, his explanation of the foundations of mathematics can be found scattered in pieces and bits in several of his writings some 50 years after he began that study, after he had been a teacher and lecturer for many years, and published only after his death in 1951 (Marion, 1998). One goal of Wittgenstein's effort was to challenge the mindset people have about mathematics, and open a door to new ways of thinking. In this essay I have extracted and put together into topical categories his expressions about mathematics that were published in his collection entitled *Zettel* (Wittgenstein, 1967). I have taken *Zettel* as a resource, and from each paragraph identified probing questions and hypothetical challenges that will give teachers a repertoire of questions and knowledge about mathematical reasoning and reflection, and suggestions on how best to stimulate mathematical explanations and justification.

This paper is intended to help teachers prepare to make meaningful discourse happen in their own classrooms. Where I have paraphrased or expanded, I have endeavored to preserve the sense and uniqueness of Wittgenstein's insightful comments. This paper is not an overview identifying which of Wittgenstein's concepts and explanations were most influential on mathematicians and in the development of mathematics both at Cambridge and more broadly around the world. Nor is it an attempt to identify what is missing in *Zettel* that might be needed to complete the discussion of the foundations of mathematics. Much of that discussion can be found in other books compiled from Wittgenstein's writings and published posthumously (Wittgenstein, 1978). I have resisted the temptation to say too much about what Wittgenstein is trying to uncover with his probing questions and aphorisms. The purpose of this paper is to gather in one place a collection of insightful questions and probes that could be used to help teachers pose problems for and with their students.

As testimony to Wittgenstein's relevance in teaching mathematics, his writing and teachings have inspired many authors and contemporary experts. Anna Sfard (2008) pays homage to both Lev S. Vygotsky and Ludwig Wittgenstein:

...[T]wo giants whose shoulders proved wide enough to accommodate legions of followers and a wide variety of interpreters ... [They] continue to inspire new ideas even as I am writing these lines. This, it seems, is due to one important feature their writings have in common: rather than provide information, they address the reader as a partner in thinking; rather than presenting a completed edifice with all the scaffolding removed,

they extend an invitation for a guided tour of the construction site; rather than present firm convictions, they share the ‘doubt that comes after belief.’ These two writers had a major impact on my thinking; I can only hope they had a similar effect on my ability to share it. (pp. xxi-xxii)

Whereas the writings of Vygotsky are widely studied by pre-service teachers and educators, Wittgenstein is widely quoted but seldom studied. I hope that this paper contributes to correcting that deficit.

Wittgenstein usually uses general or surrogate subjects (i.e. colors, words) rather than discipline specific topics as examples, and it is up to the reader to see and say “Aha... that also explains what is at the basis of mathematics.” Some of the following quotes are to be viewed in this way; others are more obviously applicable to the discussion of the foundations of mathematics. I have laid these side by side (quotes from Wittgenstein are in italics) to invite development of a provocative discussion about the foundations of mathematics, as well as to prompt provocative questions for teachers to use productively when planning and implementing instruction. After a brief introduction and historical background, what follows is organized into eight categories - Emerging Potentialities; Practice Applying Rules; Origins in Nature; Mathematical Procedures; Cultural Implications; Teaching relationships; Familiarity with Proofs; and Philosophy of Mathematics - to provide a source of questions that are perhaps most germane to the reader’s present teaching needs.

### ***Emerging Potentialities***

Important mathematical understanding does not begin in the teen years when students are for the first time exposed to Euclidean Geometry and proofs beginning with definitions and axioms. Long before this, playtime experimentation with objects and quantities prepares each child’s mind to receive more formal instruction in the abstractions that we know as numbers, counting, arithmetic and all parts of mathematics. But children are seldom encouraged to reflect on the progression from play to formal mathematical reasoning.

*700. Why do we count? Has it proved practical? Do we have the concepts we have, e.g. our psychological concepts, because it has proved advantageous? –And yet we do have certain concepts on that account, we have introduced them on that account.*

*701. At any rate the difference between what are called propositions of mathematics and empirical propositions comes to light if one reflects whether it makes sense to say: ‘I wish twice two were five!’*

With these questions, Wittgenstein prompts us to think about what is special about mathematics. When scientists wish, they develop hypotheses; when writers wish, they develop fiction; when doctors wish, they look for alternative cures. In mathematics, wishing is akin to conjecturing, a playfulness in asking “What if?”, and “Why?”, and “Why not?”, and “Why does it have to be this way?” Taking these questions as a beginning point, I will show how Wittgenstein builds an explanation of the language-game of mathematics and how it is not only a different (but in important ways parallel) activity to the way we think and live. This language-game<sup>2</sup> is a form of

---

<sup>2</sup> Mathematicians often create their own vocabulary by giving special meanings to ordinary terms and phrases. Giving specialized meaning to old terms allows mathematicians to say things that might otherwise be difficult to say. But this can be confusing to novices who assume the “ordinary language” meaning.

being and doing that is interlaced with our language and our daily activities. As such, it is also an intellectual feature of our lives that is identifiable as a separate, dynamic activity.

A historical perspective can be useful, and here Wittgenstein suggests reflection on one's own personal history as it relates to things mathematical. Questions teachers might ask students: How did you learn to count? When did you begin to do arithmetic? Can you rephrase this (any particular) discussion into a mathematical proposition? Why is this a mathematical proposition? Can you wish it to be different, or is it somehow determinate? Can you wish a certain kind of result? Can you influence the result with your will? Do you take what you get?

By the time a child develops the notion of *conservation of numbers* the ability to think and talk mathematically is normally well underway (Piaget, 1952.) Researchers have identified rudimentary mathematical thinking in babies as early as two months old (Dehaene, 1997) which suggests that the basic sense about "number" is inherited and perhaps instinctive in humans. Elementary problem solving, a child's propensity to experiment and make quantitative choices, can be witnessed very early by all observant parents.

103. ...--But what should be called 'making trials' and 'comparisons' can in turn be explained only by giving examples, and these examples will be taken from our life or from a life that is like ours.

104. If he has made some combination in play or by accident and he now uses it as a method of doing this and that, we shall say he thinks. --In considering he would mentally review ways and means. But to do this he must already have some in stock. Thinking gives him the possibility of perfecting his methods. Or rather: He 'thinks' when, in a definite kind of way, he perfects a method he has. [Marginal note: What does the search look like?]

105. It could also be said that a man thinks when he learns in a particular way.

Our shared childhood cultural development prepares us to develop (or inhibits) our capacity to deal with, think about, and operate with numbers and mathematical symbols. Our intuition (or considering, as Wittgenstein describes it), guides us. Educators can learn to encourage its development in students and teachers.

Questions: How can you make a conjecture and use your knowledge to verify your claim? How might you pose "what if?" questions? How can you vary the parameters or values to see a generalizable result? What personal experience have you had that shows you that this result is valid? Can you predict the result of your work by estimating, rounding off and achieving a quick guess? How does your answer compare to the estimate? What do you think will happen if ...?

And of course counting and speaking with numbers has a long tradition even in many prehistoric cultures. For example, the Ishango bone, a piece of bone notched with what are believed to be mathematically significant tally marks, was discovered in the small African fishing village of Ishango, on the border of Zaire and Uganda. It is one of the oldest known mathematical artifacts, dating to 20,000 BC. (Zaslavsky, 1979).

143. We might say: in all cases what one means by 'thought' is what is alive in the sentence. That without which it is dead, a mere sequence of sounds [i.e., mathematical symbols] or written shapes...

*144. How words [or symbols] are understood is not told by words alone [explanation of meaning].*

This suggests not just a vital, but essential role for pictures, graphs, images, symbols, manipulatives and other visual displays in mathematics that help to explain and illuminate mathematical ideas.

There were well developed cultures in California such as Yahi (Kroeber, 1976) from which we have first-hand accounts of how numbers and mathematical thinking were passed on as a sacred, mystical tradition comparable to the way some contemporary people use Numerology. For most of us mathematics is the language of science and is interwoven into our mundane and routine lives. But superstitions surrounding and containing numbers persist in many cultures, including our own.

Questions: Why is 3 a lucky number? Why is thirteen considered an unlucky number in many cultures? Why is 4 an unlucky number in Japan? Can you give other examples of how number words are used in superstitious ways, or, of how numbers are used in popular culture, and imbued with special meaning? What is special about prime numbers? Or perfect numbers?

Mathematics teachers may find it interesting and rewarding to watch students in the beginning stages as they progress through ever more complex applications. They go through false starts of understanding, and then go back and review until it is completely integrated into their mental framework. Even the average student can go back to the beginning of the book half way through the course and do the “easier” mathematics with greater facility than during the first days. Although many students seem to struggle, they learn something, about both the content and the tacit rules of operation and construction.

*153. It somehow worries us that the thought in a sentence is not wholly present at any one moment. We regard it as an object which we are making and have never got all there, for no sooner does one part appear than another vanishes.*

This is the kind of mental thinking that can take place as students learn mathematics, or as someone develops mathematics even in the most fundamental settings. It helps to explain why not all students do their homework correctly each day. Understanding is elusive and sometimes tenuous.

Questions: How can you repeat this and say it in a different way? How do we incorporate knowledge of irrational numbers, imaginary numbers and exponents, for example, as part of our operational system? What rule should we use to factor a polynomial?

I invite you to try Wittgenstein’s exercise which relates to the development of mathematics:

*310. Imagine you had to describe how humans learn to count (in the decimal system, for example). You describe what the teacher says and does and how the pupil reacts to it. What the teacher says and does will include e.g., words and gestures which are supposed to encourage the pupil to continue a sequence; and also expressions such as ‘now he can count’. Now should the description which I give of the process of teaching and learning include, in addition to the teacher’s words, my own judgment: the pupil can count now,*

*or: now the pupil has understood the numeral system? If I do not include such a judgment in the description –is it incomplete: And if I do include it, am I going beyond pure description: --Can I refrain from that judgment, giving as my ground: 'That is all that happens'?*

Thus, do we think that the evidence for learning is just the accomplishment of the task, not the process developed, or the explanation that is given? Can he/she still do it next week? How does that relate to participation in the classroom, when a few bright students consistently raise their hands first, or dominate in group activity? We should encourage the bright students, give them praise; but should we give only them the extra challenge of justifying, proving or generalizing? Or is it better to give them an additional application?

Questions: That looks excellent, tell us how you did that? Can you justify your work? How might you generalize your results? Is there another way to look at this problem?

### ***Practice in Applying Rules***

Yet the words for numbers and operations that match symbols, and our facility with these at ever more sophisticated levels, have their origin in the way we are taught.

*145...If the sign is an order, we translate it into action by means of rules [e.g., + for addition, or] or tables. It does not get as far as an impression, like that of a picture; nor are stories written in this language.*

*146. In this case one might say: 'Only in the system has the sign any life.'*

Mathematics is in many ways a different language, woven into our ordinary language (or form of life) that some master and with which others struggle for their entire lives. It is easy to imagine telling someone something about the measurement of a shirt or a hat using mathematical terms and realizing by looking at the expression on their face, or hearing their questions, that they do not have facility with the mathematics being used, that they are outside the system, do not feel at ease participating in that language game.

Questions: Why do we use Greek letters as symbols in mathematics? Can you write a sentence that restates what this sign signifies, what it is telling you to do? What is an algorithm? Can you identify and emphasize the key vocabulary words (in a new mathematical concept)? Who developed this symbol, and when did it come into popular mathematical vernacular? What problems were being posed and why was it important to solve them? When do we use this word to mean something else? And is this the best word you can choose for this mathematical use?

Wittgenstein gives a story of how people develop and use rules which provides some insight into the foundations of mathematics and their transfer to new generations.

*292. ...If an order were given us in code with the key for translating it into English, we might call the procedure of constructing the English form of the order 'derivation of what we have to do from the code' or 'derivation of what executing the order is'. If on the other hand we act according to the order, obey it, here too in certain cases one may speak of derivation of the execution.*

*293. I give the rules of a game [e.g., matrix algebra]. The other party makes a move, perfectly in accord with the rules, whose possibility I had not foreseen, and which spoils*

*the game, that is, as I had wanted it to be. I now have to say: 'I gave bad rules; I must change or perhaps add to my rules.'*

*So in this way have I a picture of the game in advance? [Sometimes yes, sometimes no – perhaps we are inventing new mathematics] In a sense: Yes.*

*It was surely possible, for example, for me not to have foreseen that a quadratic equation need have no real root.*

When Wittgenstein uses examples from mathematics that help explain his point it is perhaps easier to see how he intended all of his discussions to also apply to the foundations of mathematics even when it is ostensibly about language or thinking. The concept of following rules is an essential ingredient of Wittgenstein's notion of the foundations of mathematics.

Questions: What rule are you following here? And why is this important to follow? Why is it important in addition to align numbers with the decimal points in a vertical column? When you find a mistake in your procedure or result, what rule have you violated? How would you explain how you can expand this compact notation? Why do you simplify improper fractions?

It seems reasonable to support the idea that teaching and training is how mathematics is transferred from generation to generation, but how did the first mathematician develop his or her techniques of solving problems such as telling time or measuring distances? Telling time using an analog clock is a rather advanced mathematical activity, judging from the many intermediate steps of knowledge development that are needed. Examining the roots of learning mathematics is useful both for its own sake and as an activity of hypothesizing about how mathematics came to be and how it developed step by step. It is now well acknowledged that mathematical knowledge is socially constructed (Sfard, 2008; Vygotsky 1962, 1978).

*412. Am I doing child psychology? –I am making a connexion(sic) between the concept of teaching and the concept of meaning.*

*413. One man is a convinced realist, another a convinced idealist and [each] teaches his children accordingly. In such an important matter as the existence or non-existence of the external world they don't want to teach their children anything wrong...*

*419. Any explanation has its foundation in training. (Educators ought to remember this.)*

Their training, their personal beliefs and competency in mathematics together determine how teachers approach the task of teaching. What determines how well students develop a meaningful understanding of mathematical concepts? If mathematics is truly a distinguishable language-game, does a thorough understanding change the person's life in a way that no other knowledge does, e.g., history?

Questions: Is the context of mathematics important to you, or do you simply enjoy engaging in the game, puzzle, or algorithm? Do you need to see real-world applications of mathematics in order for it to make sense and be meaningful and useful to you? Would you learn mathematics more easily if it were connected to your daily life and activities? Write a short story to illustrate this mathematics.

Clearly there are a lot of linguistic and cultural skills that come into play when we set out to teach and learn mathematics. Consider this simple act of mimicking a certain procedure:

305. *'Do the same.'* But in saying this I must point to the rule. So its application must already have been learnt. For otherwise what meaning will its expression have for him?

306. To guess the meaning of a rule, to grasp it intuitively, could surely mean nothing but: to guess its application. And that can't now mean: to guess the kind of application; the rule for it. Nor does guessing come in here.

Students who guess about rather than understand the structure of the application of the rules of mathematics often commit the error of false generalization; they apply rules to algorithms where these do not apply. And they end up believing that mathematics is akin to magic, or at least mysteriously complicated.

Questions: Why can you do the same in this case? Why can you apply the same algorithm here? Are you able to transfer your knowledge to a new situation? Can you generalize? Can you extend your result? Try to look at different ways of adding and multiplying fractions, and see the results obtained when these rules are applied differently. How can you decide whether this alternative algorithm is valid?

It is obvious that at each step along the way rules and applications have to be developed, assessed, revisited, refreshed, refined and verified before they can be fully understood.

307. *...The application of a rule can be guessed only when there is already a choice between different applications.*

This statement points to the prior experience and cultural learning from which the student benefits but which is often taken for granted. If this is missing then the teacher has to supply some kind of equivalent explanation or experience.

308. *We might in that case also imagine that, instead of 'guessing the application of the rule,' he invents it. Well, what would that look like? Ought he perhaps to say 'Following the rule +1 may mean writing 1, 1+1, 1+1+1, and so on'? But what does he mean by that? For the 'and so on' presupposes that one has already mastered a technique.*

Here Wittgenstein is most explicit about discussing the foundations of mathematics, that these are based on the idea that "...one has already mastered a technique." This is in part what it means when we say that mathematical knowledge is socially constructed. A good deal of mathematical technique is imbedded in our mathematical language. It is developed as tacit knowledge as well, but in learning mathematics we take it out, organize it and put it into a new language-game of its own called doing mathematics. When we begin geometry we already have ideas about a point and a line, but learn new concepts, and also new definitions that explain and clarify familiar concepts.

Questions: Where did the concept of zero develop, and why? What would life be like without negative numbers? How did you know what to do to find the next and  $n^{\text{th}}$  term in a sequence? What does  $n$  stand for?

### **Origins in Nature**

The suggestion that mathematics is merely a collection of rules to be applied, rather than truths of nature, however, is disputed by most mathematicians who think mathematics is fundamentally about patterns, and is taken from nature.

*293 (cont.) Thus the rule leads me to something of which I say: 'I did not expect this pattern: I imagined a solution always like this...'*

And thus our facility with mathematics grows as does the field of mathematics itself.

*294. In one case we make a move in an existent game, in the other we establish a rule of the game. Moving a piece could be conceived in these two ways: as a paradigm for the future moves, or as a move in an actual game.*

Since the debate about whether mathematics is discovered or invented is long-standing, when does it make sense to suggest that Plato with his 'forms' was correct? The fact that mathematics is essentially impervious to this reflective debate simply goes to show how knowledge of mathematics leads (one way or the other) to an identifiably separate form of life.

Questions: Take me step by step through the process you used to get that result. Is it possible that some mathematics is both discovered in nature and some invented by humans? Do the processes you used and your results hold for any such mathematical proposition...? Can you generalize? Can you justify this? Can you restate the problem for me in your own words? Can you write a proof?

At the very beginning stages of developing the capacity to do mathematics our training takes many forms. But one must look at Wittgenstein's use of the word "training" in its broadest sense, and not just as if one set out to train a horse. This training might include activities that Wittgenstein describes in this somewhat different context of learning language. Through this "training" we discover that mathematics connects to and explains reality; it is not fiction or about arbitrary shapes; further it involves repetition, trial and error and learning by physical manipulations.

*195. Let us imagine a kind of puzzle picture: there is not one particular object to find; at first glance it appears to us as a jumble of meaningless lines, [or an undecipherable mathematical formula or a tricky story problem] and only after some effort do we see it as, say, a picture of a landscape. –What makes the difference between the look of the picture before and after the solution? It is clear that we see it differently the two times. But what does it amount to to say that after the solution the picture means something to us, whereas it meant nothing before?*

Mathematics often organizes and gives order to nature. This principle of pattern recognition is the basis for repetition in teaching, as well as being basic to discovery learning in various forms. This repetitive process is said to lead to "endorsable" mathematical statements when by following well-defined rules anyone is able to come to the same result (Sfard, 2008; Lakatos, 1976).

Questions: How does this relate to your personal experience? Have you seen or done this before? How does this connect with your prior knowledge? Given a sequence of numbers, how do you find the next number? What is your strategy? How is learning mathematics analogous to and different from learning a language? Can you apply this same rule in a problem that has bigger



numbers, more complications, symbols instead of numbers? At what point did you figure it out, or gain understanding? Did it just suddenly make sense?

Is the result obtained in §195 based on previous rehearsal, or training? Have you acquired the system or algorithm in your mental framework, where the signs and numbers have useful and specific meanings?

*228. Explain to someone what the position of the clock-hands that you have just noted down is supposed to mean: the hands of this clock are now in this position. –The awkwardness of the sign in getting its meaning across, like a dumb person who uses all sorts of suggestive gestures –this disappears when we know that it all depends on the system to which the sign belongs. We wanted to say: only the thought can say it, not the sign.*

Wittgenstein often tries to tease his reader into what might be thought of as a traditional mental cramp i.e. “...only thought can say it...” This explains how his writings become misunderstood when one mistakes his pedagogy for doctrine. He simply wants to say that there is a good deal of fundamental cultural background and training that gets absorbed prior to one’s learning to tell time, for example.

Questions: From where did you get your concept of time passing? What other kinds of clocks do you know about? How can other mechanisms be used to create and communicate [i.e. spherical geometry] information? How do signs get their meanings i.e.  $\div$ ,  $\times$ ,  $\infty$ ? In what other part of life do signs have special meanings? Describe the different lengths of hands on the clock and how they are used for different kinds of time. When did you first learn this? Who first developed time as a concept? Why is time so important? Why are there 24 hours in a day?

### **Mathematical Procedures**

What does it take to know we have solved a problem correctly, followed a rule, identified a characteristic of nature, or developed a useful model? This is certainly an important detail:

*196. We can also put this question like this: What is the general mark of the solution’s having been found?*

When faced with problems and challenges of increasing complexity, whether caused by curiosity or necessity, humankind did develop counting, geometry and mathematical techniques for solving problems, over periods of thousands of years in several geographically isolated locations around the globe. We spoke earlier of endorserable results to mathematical theorems and propositions. This discussion of verification and justification progresses from this into a usable and repeatable procedure. As mathematics grows, this takes us well beyond the exposition of the foundations of mathematics.

Questions: What shall we take as proper or sufficient explanation, justification? At each level of mathematics what constitutes a proof? What preparation gives students a better capability to do detailed proofs later? Can you take the solution and put it back into the original problem to verify it? What constitutes thinking mathematically?

### **Cultural Implications**

There have been many studies that have shown how success in learning mathematics can be culturally biased. We criticize standardized tests for having cultural bias that can interfere with students' understanding of the input, the instructional language, as well as the elements of a problem, making it difficult for students to demonstrate their mathematical knowledge.

*201. For someone who has no knowledge of such things a diagram representing the inside of a radio receiver will be a jumble of meaningless lines. But if he is acquainted with the apparatus and its function, that drawing will be a significant picture for him.*

Having a broad frame of reference helps students learn mathematics and mathematics contributes to this cultural growth of students. Once a method to solve a problem was developed, it became part of an expanding form of life that was often shared in an "interactive sphere." The history of  $\square$  gives a clue to how ubiquitous and specialized mathematical thinking has been. We have to also learn (and we share) verification skills that are practical and applicable in our lives as we progress with our mathematical education..

Questions: Does that hold true every time you do this...? For every number? What does it mean to use sample data that are representative of the whole population? How can you extrapolate your results? What kind of visualization skills can you employ to see this shape differently? Is mathematics a "universal" language across national borders? Can you verify your result best with a sketch, graph, or diagram?

The skill of verification, and motivation for it, is fundamental to the development of the language-game of mathematics. Developing this is also connected to developing socio-mathematical norms in the classroom, and with practice these will become a matter of course.

*309. We copy the numeral from 1 to 100, say, and this is the way we infer, think.*

*I might put it this way: If I copy the numerals from 1 to 100 –how do I know that I shall get a series of numerals that is right when I count them? And here what is a check on what? Or how am I to describe the important empirical fact here? Am I to say experience teaches that I always count the same way? Or that none of the numerals gets lost in copying? Or that the numerals remain on the paper as they are, even when I don't watch them? Or all these facts? Or am I to say that we simply don't get into difficulties? Or that almost always everything seems all right to us?*

*This is how we think. This is how we act. This is how we talk about it.*

This is how the foundations of mathematical verification skills get sorted out, often by being told: "Do this, don't do that." Or by self discovery. When students learn to contribute their own reasoning and justifications they take ownership of the mathematics they are learning, and this new language becomes part of who they are.

Questions: How can you do that a different way? Can you explain your work step by step? How can you look at this operation and find the mistake or error? What role does neatness have in preventing errors and in developing a clear form of communication?

There is the expectation, unspoken in most cases, that we use mathematics to connect to, understand and explain the world around us, not only as an abstract or academic activity. We

mathematize our world in so many ways. (Sfard, 2008) How fast is the wind blowing? How much food (how many calories) should we (or do we) eat in one sitting? We celebrate birthdays and count the days to the next major holiday. We score competitive events, compile performance statistics, and race the clock to meet deadlines. Our whole economy is built on the use of coinage, and measured in billions and trillions. It is no secret and no coincidence that mathematics has become the most widely spoken and only “universal” language of humanity. Someone who is ignorant of mathematics gets left out at some level.

*695. The understanding of a mathematical question. How do we know if we understand a mathematical question?*

*A question –it may be said—is a commission. And understanding a commission means: knowing what one has got to do. Naturally, a commission can be quite vague – e.g., if I say ‘Bring him something that’ll do him good’. But that may mean: think about him, about his state etc. in a friendly way and then bring him something corresponding to your sentiment towards him.*

*696. A mathematical question is a challenge. And we might say: it makes sense, if it spurs us on to some mathematical activity. [E.g., making change, buying groceries.]*

*697. We might then also say that a question in mathematics makes sense if it stimulates the mathematical imagination.*

*698. Translating from one language into another is a mathematical task, and the translation of a lyrical poem, for example, into a foreign language is quite analogous to a mathematical problem. For one may well frame the problem ‘How is this joke (e.g.) to be translated (i.e. replaced) by a joke in the other language?’ and this problem can be solved; but there was no systematic method of solving it.*

When we translate our activities, patterns or problems from the real world into mathematical operations, how well have we done this? (Mathematical modeling and applications.)

Questions: Does it work and make some sense compared to what we are looking at...? Does this make sense, have we noticed, interpreted and translated reality accurately? Can you unpack this definition into plain language? Can you give an example? What is the proof all about? What does it say to us? What if you change this... ?

Teachers have a tendency to think in learned patterns, avoiding the assumptions inside, below or prior to these theories or customary operations. Is it helpful to know that much of mathematics and learning about mathematics is connected to our culture and to the thinking processes we have developed? Is this a benefit or a hindrance?

*375. These are the fixed rails along which all our thinking runs, and so our judgment and action goes according to them too.*

*382. In philosophizing we may not terminate a disease of thought. It must run its natural course, and slow cure is all important. (That is why mathematicians are such bad philosophers.)*

Wittgenstien’s humor and biases aside, teachers and learners do have to expect to develop new skills apart from the way they learned mathematics. If we just keep doing it the same way, pedagogy will not improve.

### **Teaching Relationships**

When operating mathematically, do we use the same mental equipment that is used for other linguistic (or cultural) activities? Is the thinking and remembering of music any different? How much does it depend on a certain propensity for logical thinking or common sense? Why are some people said to be “mathematically inclined?” i.e. Gardner’s measure of Logical/Mathematical (multiple) intelligence? (Gardner, 2003)

666. *I can display my good memory to someone else and also to myself. I can subject myself to an examination. (Vocabulary, dates) [And in mathematics.]*

667. *But how do I give myself an exhibition of remembering? [is knowing different?] Well, I ask myself ‘How did I spend this morning?’ and give myself an answer. –But what have I really exhibited to myself? Remembering? That is, what it’s like to remember something? – Should I have exhibited remembering to someone else by doing that?*

668. *Forgetting the meaning of a word [or of a symbol] and then remembering it again. What sorts of processes go on here? What does one remember, what occurs to one, when one recalls what the French word ‘peut-etre’ means?*

669. *If I am asked ‘Do you know the ABC?’ and I answer “Yes’ [Can you count to 20?] I am not saying that I am now going through the ABC in my mind, or that I am in a special mental state that is somehow equivalent to the recitation of the ABC.*

672. *...This calculation takes one and a half minutes; but how long does being able to do it take? And if you can do it for an hour, do you keep on starting afresh?*

Here Wittgenstein is again trying to tease the reader into a mental cramp, or theory to answer these hypothetical quandaries e.g. “...how long does being able to do it take?” Scientists now know that different parts of the brain are involved in different tasks, but all of those parts are components, interconnected and at the disposal of our total thinking apparatus. Similar elements and considerations are involved in doing mathematics that are identifiable in our ordinary language i.e. grammar, syntax, iconography, vocabulary... and the mental activities that are engaged flow together and we usually just take these for granted. Experienced teachers and many text books provide learning strategies, and It is a useful teaching practice to solicit strategies from students and make learning strategies explicit.

Questions: Why is it important to learn your multiplication facts until you own them? What is math fluency? Why is it helpful to be able to convert certain key fractions to percentages from memory? Does speed and accuracy in doing the algorithms of arithmetic predict future math competency? What is the best way to learn this procedure so that you will remember it? How do you remember what to do with this (or write this) proof? Why is it important for you to have a clear understanding of definitions, axioms, and be able to apply them in proving theorems?

By now it should be clear that a facility with arithmetic and with all other mathematics requires training and motivation to learn.

355. *...If we teach a human being such-and-such a technique by means of examples, -- that he then proceeds like this and not like that in a particular new case, or that in this case he gets stuck, and thus that this and not that is the ‘natural’ continuation for him: this of itself is an extremely important fact of nature.*

How do we develop and expand our logical-mathematical skills? It is important in teaching to understand proximal levels of knowledge. This is also important in the discussion of developing proofs. Proving in a subtle way involves a meta-language and techniques that are outside the operation of the mathematics in question, and yet connected as a “new case.”

Questions: Can you give me an example of ... from your experience? How did you solve this challenging problem? Explain how you thought about it. Can you draw a picture of this? How do we develop visualization skills? Can you check this conjecture with several examples to determine whether it is likely true or not?

Wittgenstein’s discussion of how mathematics is usually a matter of accepted and learned conventions, takes on a subtle touch when he uses an example involving taste.

*366. Confusion of tastes: I say ‘This is sweet’, someone else ‘This is sour’ and so on. So someone comes along and says: ‘you have none of you any idea what you are talking about. You no longer know at all what you once called a taste.’ What would be the sign of our still knowing? (Connects with a question about confusion in calculating.)*

*367. But might we not play a language-game even in this ‘confusion’? – But is it still the earlier one?—*

Teachers know that there is often more than one way to solve a problem.

*373. Concepts other than [those] akin to ours might seem very queer to us [e.g., performing arithmetic in base 8]; deviations from the usual in an unusual direction.*

Questions: How many different ways can you write a statement of division? Can you look at this alternative way of solving this problem and decide whether it is valid? You can visualize when we graph in two and three dimensions, but what happens in your mind when you have to use four and more dimensions in a problem? Can you count and perform arithmetic in base 8, or use numbers to write in code? Is mathematics a science? What did you think about as you solved that problem?

Teachers might suggest to students to read over their homework assignment when they receive it, make sure they understand it all, and just let it ferment in the brain for a few days. In this way, the problem runs its course and a solution often appears unbidden. What does it mean to know mathematics? Why is scientific notation useful?

*387. I want to say: an education quite different from ours might also be the foundation for quite different concepts.*

*388. For here life would run on differently. –What interests us would not interest them. Here different concepts would no longer be unimaginable. In fact, this is the only way in which essentially different concepts are imaginable.*

Textbooks change and often different textbooks use different notation or different statements of the same theorems, definitions, and concepts. Is it possible to imagine a different form of mathematics in the same way one could imagine using different number bases?

Questions: Does deciphering codes involve using special techniques? How does guessing contribute to solving mathematical problems? How does the development of the computer give rise to a whole different kind of mathematics previously unimaginable? Why did you solve it this

way? Can there be different geometries? IS the mathematics of infinity the same as regular mathematics?

In learning the fundamentals of mathematics, one comes to know early on that symbols are specific, and learns how to use each of them in so many different circumstances, and the list grows with each new problem solved.

333. *'Red is something specific' –that would have to mean the same as: 'That is something specific' –said while pointing to something red. But for that to be intelligible, one would have already to mean our concept of 'red', to mean the use of that sample.*

Substitute the equal sign '=' for "red" in this sentence, and we see how understanding the operation of a simple symbol has at its root some understanding of abstractions and the use of symbols and the '...use of that sample.'

334. *I can indeed obviously express an expectation at one time by the words 'I'm expecting a red circle' and at another by putting a coloured(sic) picture of a red circle in the place of the last few words. But in this expression there are not two things corresponding to the two separate words 'red' and 'circle'. So the expression in the second language is of a completely different kind.*

When we think of the symbol  $\square$  it has a meaning not only because of what we have otherwise or previously learned about it, so we can use it without reinventing it every time, but also on a deeper or more fundamental level associated with our ability and capacity to use symbols at all:

336. *...The important question here is never: how does he know what to abstract from? but: how is this possible at all? or: what does it mean?*

Questions: What does the number one '1' stand for? Are numbers abstractions in the same way that other symbols are? Or in a different way? Which symbols are representative such as "£" or " $\Sigma$ "? Which contain rules or instructions?

In reading Wittgenstein it is easy to get caught up in the actual content of his examples, rather than retaining the idea that color, for example, is a surrogate to develop the philosophical concept he is driving at. Here he reaffirms this idea.

347. *The fact that we calculate with certain concepts and not with others only shews(sic) how various in kind conceptual tools are (how little reason we have here ever to assume uniformity). [Marginal Note: On propositions about colours that are like mathematical ones e.g. Blue is darker than white. On this Goethe's Theory of Colour.]*

Here one might reflect on the lack of uniformity between using cardinal versus ordinal numbers, or the multiplicity of ways of representing division. And calculations are performed using cardinal numbers, 1, 2, 3, 4..., not 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>..., but also the letters, x, y, n... are used and are fundamental to mathematics.

Questions: How can color be important in developing mathematical concepts? Why is red used to show a deficit or a loss? How effective are Roman numerals for mathematical manipulations?

Once the process of learning and reproducing proofs begins, mathematics is viewed in a somehow different way, as from the outside looking in, as suggested, with ordinary academic 'input language' as a meta-language. Concepts such as generalization and justification become recognized as important.

*171. But isn't understanding shewn(sic) e.g. in the expression with which someone reads the poem, sings the tune? [or completes the math problem?] Certainly, but what is the experience during the reading? About that you would just have to say: you enjoy and understand it if you hear it well read, or feel it well read in your speech-organs.*

*172. Understanding a musical phrase [or proof] may also be called understanding a language.*

This kind of aesthetic is often related to observing the quality and elegance of a well written proof. And being able to write the proof and derive it, shows an understanding well beyond simple performance or even beyond explanation. How do teachers share their interest and passion for mathematics with their students?

Questions: What are the different techniques you use to write proofs? When do you need to provide a proof of a mathematical operation or formulation?

Is understanding mathematically akin to “understanding a musical phrase” or just another move in a specialized language-game?

### ***Familiarity with Proofs***

But still students often struggle and lose the meaning of some of the terms, or use them in the wrong way, or misunderstand their instructions or the prescribed techniques.

*183. The man I call meaning-blind will understand the instruction ‘Tell him he is to go to the bank—I mean the river bank,’ but not ‘Say the word bank and mean the bank of a river’. What concerns this investigation is not the forms of mental defect that are found among men; but the possibility of such forms. We are interested, not in whether there are men incapable of a thought of the type: ‘I was then going to ...’ –but in how the concept of such a defect should be worked out.*

*If you assume that someone cannot do this, how about that? Are you supposing he can't do that either? –Where does this concept take us? For what we have here are of course paradigms.*

How do students understand the expressions the teacher of a proofs class uses? What is the prospect for discourse in finding meaning through conversation and exchange of ideas among students? It seems that all one's skills with language and mathematics go into developing an acceptable proof; it all comes together in the mind with the suggested rules and ‘paradigms’ or it doesn't. And for so many people; why doesn't it?

Questions: Do you learn mathematics better in your first language? When should you use mathematical induction? When contradiction? What is the role of definitions? When you don't follow instructions, does that mean you can't do the mathematics next time? How do you learn to write a mathematical proof?

In a certain way that is not always clear to a teacher, the practice of and reasons for doing proofs are quite foreign to some students. And of course over the years the requirements of what it takes to make a good proof have changed; certainly Euclid's explanations have been revised and improved a little. The standards of proof have changed; the conclusions are sometimes different because the postulates are different. The discovery of non-Euclidean geometry throws a whole new set of wrinkles into the discussion.

393. *It is easy to imagine and work out in full detail events which, if they actually came about, would throw us out in all our judgments.*

*If I were sometime to see quite new surroundings from my window instead of the long familiar ones, if things, humans and animals were to behave as they never did before, then I should say something like 'I have gone mad'; but that would merely be an expression of giving up the attempt to know my way about. And the same thing might befall me in mathematics. It might e.g. seem as if I kept on making mistakes in calculating, so that no answer seemed reliable to me.*

*But the important thing about this for me is that there isn't any sharp line between such a condition and the normal one.*

This kind of imagined dissonance can make students dislike mathematics. There are special procedures to use when solving problems (or when attempting to write proofs); once these are learned there will be more tools available for solving some problems, but that is not always sufficient to allow generalizations to be made to other problems.

Questions: Explain the steps you took; what was your thinking? Why do you think your solution is correct and makes sense? What will you do differently next time?

If students fail to learn and do proofs, is it because they are not fully conversant or fluent with the language-game of mathematics?

185. *It's just like the way some people do not understand the question 'What colour has the vowel a for you?' –If someone did not understand this, if he were to declare it was nonsense—could we say he did not understand English, or the meaning of the individual words 'colour', 'vowel' etc.?*

*On the contrary: Once he has learned to understand these words, then it is possible for him to react to such questions 'with understanding' or 'without understanding'.*

And to rephrase: Once one has learned to understand a certain proof and the strategy at stake, then it is possible to react with understanding and give explanations, show competence by recognizing when a statement about the proof (or an incorrect or incomplete proof) has been given incorrectly. We have thus mastered the language-game.

186. *Misunderstanding –non-understanding. Understanding is effected [and affected] by explanation; but also by training.*

It can be a valuable teaching technique to give students incorrect or invalid proofs, and have them identify what is wrong, or distinguish valid from invalid proofs. Such exercises are useful pedagogical tools not only to assess the students' grasp of the proof and the mathematics in question, but also to help students learn the language of proof writing.

Questions: What do you mean when you use these words...? Does this necessarily follow? Is this always true? What are the applications of this?

Is it always possible to show reasoning, or is intuition a valuable asset in doing mathematics? The essential motivation for doing and understanding proofs is to accept the quality of 'knowing' that is fundamental to the language-game of mathematics.



408. *But isn't there a phenomenon of knowing, as it were quite apart from the sense of the phrase 'I know'? Is it not remarkable that a man can know something, can as it were have the fact within him? –But that is a wrong picture.—For, it is said, it's only knowledge if things really are as he says. But that is not enough. It mustn't be just an accident that they are. For he has got to know that he knows: for knowing is a state of his own mind; he cannot be in doubt or error about it –apart from some special sort of blindness. If then knowledge that things are so is only knowledge if they really are so; and if knowledge is in him so that he cannot go wrong about whether it is knowledge; in that case, then, he is also infallible about things being so, just as he knows his knowledge; and so the fact which he knows must be within him just like the knowledge.*

*And this does indeed point to one kind of use for 'I know'. "I know that it is so" then means: I know that it is so', then means: It is so, or else I'm crazy.*

*So: when I say, without lying: 'I know that it is so', then only through a special sort of blindness can I be wrong.*

This is the kind of certainty and conviction that can be looked for in the language-game of mathematics with respect to proofs.

Questions: Does this mathematics fit and connect in some important way to reality? Does your process follow all the well established rules for arithmetic, factoring, algebra, induction, logic, contradiction or deduction etc.? Can the solutions be verified?

The proper application and execution of proofs, algorithms, and formulas ought to have a certain kind of elegance and grace, as well as a certain degree of convincing.

410. *A person can doubt only if he has learnt certain things; as he can miscalculate only if he has learnt to calculate. In that case it is indeed involuntary.*

If one were making mistakes on purpose, that is like lying or fraud in other contexts.

Questions: When is it okay to say this, do this, conclude this? When does that statement have meaning or relevance? What kind of explanation would it take to convince you that this proposition is true?

### ***Philosophy of Mathematics***

The processes and the mental activities, the forms of life, are what are fundamental to mathematics, in a way of speaking, because these are what give rise to the relevance, development and perpetuation of mathematics.

702. *If one considers that  $2+2=4$  is a proof of the proposition 'there are even numbers', one sees how loosely the word 'proof' is used here. The proposition 'there are even numbers' is supposed to proceed from the equation  $2+2=4$ ! –And what is the proof of the existence of prime numbers? –The method of reduction to prime factors. But in this method nothing is said, not even about 'prime numbers'.*

703. *'To understand sums in the elementary school the children would have to be important philosophers; failing that, they need practice'.*

In this case Wittgenstein seems to be making the argument that it is through practice and learning definitions, that we learn about even numbers and prime numbers rather than through some kind of proof, and certainly this is the normal sequence of events. We learn to play, invent, and conjecture with prime numbers, even numbers, long before we understand their significance. This kind of proof would come possibly in a graduate school mathematics classroom, not in the

first grade. We are not born with language or mathematics, or if we were we would all speak the same language.

Questions: Can you say that in a different way? How do you know you are right?

The following ending comments in *Zettel* are not just miscellany; rather these serve as interesting challenge questions about how one comes to understand the foundations of mathematics. Initially mathematics is a practical tool, producing clear and correct answers to quantitative real-world questions; however in the processes of proving and generalizing, the context is often removed, and the mathematics is used in what might seem to be an unusual way. Both activities, applications and proofs, are parts of mathematics.

704. *Russell and Frege take concepts as, as it were, properties of things. But it is very unnatural to take the words man, tree, treatise, circle, as properties of a substrate.*

Bertrand Russell admitted the limitations of his own analysis in his final comment: “After some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable...” (Russell).

Questions: Who was the first person to use this... math this way? What was Descartes’ most important contribution?

Being conversant in the history of mathematics is usually emphasized late in the learning process for want of time. Certainly this can be enrichment for any unit in mathematics. Positing a problem or conjecture, then providing the answer in the form of a summary of an historical solution takes mathematics beyond the walls of the traditional calculation-based classroom.

705. *Dirichlet’s conception of a function is only possible where it does not seek to express an infinite rule by a list, for there is no such thing as an infinite list.*

706. *Numbers are not fundamental to mathematics.*

Then what is mathematics? Is it our intellectual nature that equips us with culture, our training, our ability to follow rules, and our linguistic capacity? These are examples of quandaries that learners at any level can be encouraged to discuss (in age appropriate ways) which will give good practice in justification and reasoning, and make their mathematics training relevant to their lives.

Questions: Read this explanation and describe why it is true or false. How would you teach or explain this to someone else? How does your tacit knowledge help you?

Research into the fundamentals of mathematics and number theory can be introduced at different stages of mathematics education, not just at the college level.

707. *The concept of the ‘order’ of the rational numbers, e.g., and of the impossibility of so ordering the irrational numbers. Compare this with what is called an ‘ordering’ of digits. Likewise the difference between the ‘co-ordination’ of one digit (or nut) with another and the ‘co-ordination’ of all whole numbers with the even numbers; etc. Everywhere distortion of concepts.*

708. *There is obviously a method of making a straight-edge. This method involves an ideal, I mean an approximation-procedure of unlimited possibility, for this very procedure is the ideal. [What does it mean to create a paradigm?]*

*Or rather: only if there is an approximation-procedure of unlimited possibility can (not must)*

*the geometry of this procedure be Euclidean.*

709. *To regard a calculation as an ornament, is also formalism, but of a good sort.*

710. *A calculation can be regarded as an ornament. A figure in a plane may fit another one or not, may be taken with other ones in various ways. If further the figure is coloured, there is a further fit according to colour. (Colour is only another dimension).*

It is also possible to pose questions that match students' abilities and talents.

Question: What is the role of aesthetics in mathematics?

Summary:

This discussion of relevant quotations from *Zettel* that relate to mathematics and the foundations of mathematics is intended to serve as a resource that may stimulate insights into how mathematics came to be, and how reasoning, discourse and thinking about proofs can be stimulated. This is not intended to be an exhaustive discussion of the foundations of mathematics, but rather a source for questioning to provoke discussions of learning strategies, justifications, verifications and insights in pedagogical settings. There is more that can be said, but all explanations must come to an end, and hopefully this is a good beginning.

## References

Note: All the numbered lines come from *Zettel*, by Ludwig Wittgenstein.

- Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.
- Gardner, H. (2003). *Frames of mind, the theory of multiple intelligences*. New York: Basic Books.
- Kroeber, T. (1976). *Ishi in two worlds: A biography of the last wild Indian in North America*. Los Angeles: University of California Press.
- Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.
- Marion, M. (1998). *Wittgenstein, finitism, and the foundations of mathematics*. Oxford: Clarendon Press.
- Piaget, J. (1952). *The origins of intelligence of the child*. London: Routledge and Kegan Paul.
- Ramsey, F. P., 1925. The foundations of mathematics. Originally published in the *Proceedings of the London Mathematical Society*, Ser. 2, 25( 5), 338-384.
- Russell, B. (1956). *Portraits from memory, and other essays*. New York: Simon and Schuster.
- Sfard, A. (2008). *Thinking as communicating, Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.
- Vygotsky, L.S. (1962). *Thought and language*. Cambridge, MA: MIT Press.
- Vygotsky, L.S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.
- Wittgenstein, L. (1978) *Remarks on the foundations of mathematics*, translated by G.E.M. Anscombe. Cambridge, MA: MIT Press.
- Wittgenstein, L. (1967). *Zettel*. G.E.M. Anscombe & G.H. von Wright, Eds. Translated by G.E.M. Anscombe. Berkeley and Los Angeles: University of California Press.
- Zaslavsky, C. (1979) *Africa counts: Number and pattern in African culture*, L. Hill.

## **A Graduate Level In-Service Teacher Education Curriculum Integrating Engineering into Science and Mathematics Contents**

*Ke Wu Norman<sup>1</sup>*

*Department of Mathematical Sciences  
The University of Montana*

*Tamara J. Moore<sup>2</sup>*

*Department of Curriculum and Instruction  
University of Minnesota, Twin Cities  
Email: [Tamara@umn.edu](mailto:Tamara@umn.edu)*

*Anne L. Kern<sup>3</sup>*

*Department of Curriculum and Instruction  
University of Idaho - Coeur d'Alene*

*Abstract: This paper presents the curriculum of a master's level in-service teacher education course that integrates engineering into mathematics and science for high school mathematics and science teachers. The curricular design of the course including learning goals, reading list, course assignment and grading rubric, and a sample of Model-Eliciting Activities (MEAs) are discussed. In addition preliminary research results on teachers' perception of engineering show that prior to taking this course, teachers' understanding of engineering mainly focused on the professions of the engineering discipline. After the participation in the course, teachers' perceptions of engineering were broadened and included the design process of engineering. The curriculum and research results shared in this paper shed light on the development of k-12 teacher training programs that integrate STEM disciplines.*

**Key words:** Engineering, teacher education, mathematics and science education

### **Introduction**

In order to maintain its global leadership, the United States needs a technically literate society and an engineering-minded workforce. There is evidence indicating that America is in need of technically savvy workers (Galvin, 2002). In recent years, companies in America have spent about \$60 billion annually on training their workers on basic skills that should have been taught at school (Galvin, 2002). On the other side, the

---

<sup>1</sup> [ke.norman@mso.umt.edu](mailto:ke.norman@mso.umt.edu)

<sup>2</sup> [Tamara@umn.edu](mailto:Tamara@umn.edu)

<sup>3</sup> [akern@uidaho.edu](mailto:akern@uidaho.edu)

poor performance in mathematics and science achievement eliminates many bright students from the ranks of scientists and engineers.

Engineering education in K-12 classrooms can provide a better understanding of the components of a technical career to more students at an earlier age. The American Society for Engineering Education (ASEE) has recently launched a significant effort to make engineering methods and ideas more accessible to students in K-12 schools (Douglas, Iversen, & Kalyandurg, 2004). Further, the ASEE deems it important that teachers have a good understanding of the nature of engineering and how to integrate engineering into their classroom practice. Teacher training programs at universities and colleges need to offer courses that provide the integrated STEM (science, technology, engineering, and mathematics) learning experiences to teachers (Norman, Kern, & Moore, 2010). A course that supports and helps science and mathematics teachers in developing deeper understanding of engineering will be beneficial to their teaching and learning.

This paper presents the curriculum of a graduate teacher education course that integrates engineering contexts into science and mathematics contents. The design of the course is the result of a collaborative effort among an engineering educator, science educator, and mathematics educator. The course was first taught at the University of Minnesota Twin Cities campus in the summer of 2007. The preliminary results regarding changes in teacher perceptions of engineering through the participation in this course and will be presented later in this paper.

## Curricular Design of the Course

### *Learning Goals and Overall Course Design*

This course for in service science and mathematics teachers integrates engineering through cooperative learning with a focus on mathematics and science content. The three-credit master's level education course occurred over a three-week period, 2.5 hours per day, five days per week. The learning goals of the course are, students will (1) define engineering and the engineering method, and describe how engineering relates to pure mathematics/ science disciplines; (2) summarize the current research on teaching math and science in context; (3) summarize and integrate pedagogies of engagement, and (4) map contextual lesson plans (existing and new) to national standards in mathematics and science disciplines.

Week one classes focused on (1) getting students introduced to engineering through definition and having an engineering professor as a guest speaker, (2) reading discussions, and (3) two hands on inquiry activities. Week two focused on engineering problem solving through the introduction and the use of Model-Eliciting Activities (MEAs) (MEAs will be discussed later in this paper) (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly, & Post, 2000; Diefes-Dux, Moor, Zawojewski, Imbrie, & Follman, 2004). The third week (the final week) of the course focused on design projects, cooperative learning theory, and hands-on activities that tied mathematics and science with

engineering. This course was taught in an environment that allowed mathematics and science teachers to build partnerships and work collaboratively on engineering design projects (Johnson, Johnson, & Smith, 1998).

### Reading List for Teachers

The course contained several readings that introduced teachers to engineering education in the K-12 classrooms with a heavy emphasis on problem solving and design. Table 1 presents the list of readings.

Table 1. Reading list

<ol style="list-style-type: none"> <li>1. Anderson-Rowland, M. R., Baker, D. R., Secola, P. M., Smiley, B. A., Evans, D. L., &amp; Middleton, J. A. (2002). <i>Integrating engineering concepts under current K-12 state and national standards</i>. Paper presented at the American Society for Engineering Education Annual Conference &amp; Exposition, Montréal, Quebec, Canada.</li> <li>2. Ayas, K. &amp; Zeniuk, N. (2001). Project-based learning: Building communities of reflective practioners. <i>Management Learning</i>, 32(1), 61-76.</li> <li>3. Bransford, J. D., Brown, A. L., &amp; Cocking, R. R. (Eds.). (1999). <i>How people learn: Brain, mind, experience, and school</i>. Washington, DC: National Academies Press. (Chapter 6, 7 and 10). Retrieved on August 25, 2009 from <a href="http://www.nap.edu/openbook.php?record_id=6160&amp;page=R1">http://www.nap.edu/openbook.php?record_id=6160&amp;page=R1</a></li> <li>4. Diefes-Dux, H. A., Moore, T. J., Zawojewski, J., Imbrie, P. K., &amp; Follman, D. (2004). <i>A framework for posing open-ended engineering problems: Model-eliciting activities</i>. Paper presented at the Frontiers in Education Conference, Savannah, GA.</li> <li>5. Douglas, J., Iversen, E., &amp; Kalyandurg, C. (2004). Engineering in the K-12 classroom: An analysis of current practices &amp; guidelines for the future. <i>A Production of the ASEE EngineeringK12 Center</i>.</li> <li>6. <i>Engineering, Go for It!</i> (old and new version magazine)</li> <li>7. Gersten, R., &amp; Baker, S. (1998). Real world use of scientific concepts: Integrating situated cognition with explicit instruction. <i>Exceptional Children</i>, 65(1), 23-35.</li> <li>8. Gravemeijer, K., &amp; Doorman, M. (1999). Context problems in realistic mathematics education: A Calculus course as an example. <i>Educational Studies in Mathematics</i>, 39(1-3), 111-129.</li> <li>9. Johnson, D. W., Johnson, R. T., &amp; Houbec, E. J. ( 1998). Five basic elements. In <i>Cooperative Learning in the Classroom</i>. Edina, MN: Interaction book Company, 1998.</li> <li>10. Knapp, M. S. (1997). Between systemic reforms and the mathematics and science classroom: The dynamics of innovation, implementation, and professional learning. <i>Review of Educational Research</i>, 67(2), 227-266.</li> <li>11. Lesh, R., Cramer, K., Doerr, H. M., Post, T., &amp; Zawojewski, J. S. (2003). Model development sequences. In R. Lesh &amp; H. Doerr (Eds.), <i>Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching</i>. Mahwah, NJ: Lawrence Erlbaum.</li> <li>12. Lesh, R., Yoon, C., &amp; Zawojewski, J. S. (2007). John Dewey revisited-Making mathmatics practical versus making practice mathematics. In R. Lesh, E. Hamilton, &amp; J. Kaput (Eds.), <i>Foundations for the Future in Mathematics Education</i>. Mahwah, NJ: Lawrence Erlbaum.</li> <li>13. Lewis, T. (2006). Design and inquiry: Bases for a accommodation between science and technology education in the curriculum. <i>Journal of research in science teaching</i>, 43(5), 255-281.</li> <li>14. Lin, E. (Summer, 2006). Cooperative learning in the sceince classroom, A new model for a new year. <i>The Science Teacher</i>.</li> <li>15. No Child Left Behind Website <a href="http://www.whitehouse.gov/news/reports/no-child-left-behind.html">http://www.whitehouse.gov/news/reports/no-child-left-behind.html</a></li> <li>16. Richards, L. &amp; Schnittka, C. (2007). <i>Engineering teaching kits: Bringing engineering design into middle schools</i>. Paper presented at American Society for Engineering Education Conference,</li> </ol>
---

- Honolulu, HI.
17. Roth, W.-M. (1992). Bridging the gap between school and real life: Toward an integration of science, mathematics, and technology in the context of authentic practice. *School Science and Mathematics*, 92(6), 307-317.
  18. Smith, K. A., Sheppard, S. D., Johnson, D. W., & Johnson, R. T. (2005). Pedagogies of engagement: Classroom-based practices. *Journal of Engineering Education*, 94(1), 87-101.
  19. Wiggins, G. (1990). The case for authentic assessment. Eric digest. Retrieved April 17, 2006, from [http://eric.ed.gov/ERICWebPortal/Home.portal?\\_nfpb=true&\\_pageLabel=RecordDetails&ERICExtSearch\\_SearchValue\\_0=ED328611&ERICExtSearch\\_SearchType\\_0=eric\\_accno&objectId=090000b801b22f5](http://eric.ed.gov/ERICWebPortal/Home.portal?_nfpb=true&_pageLabel=RecordDetails&ERICExtSearch_SearchValue_0=ED328611&ERICExtSearch_SearchType_0=eric_accno&objectId=090000b801b22f5)
  20. Zemelman, S., Daniels, D., & Hyde, A. (1998). (Chapter 4, 5, and 10) Best practice in mathematics. In (2nd ed., pp.83-106) *Best Practice: New standards for teaching and learning in America's schools*. Portsmouth, NH: Heinemann.

### Homework Assignment

The grade for the course was determined through the completion of four components:

1. Annotated Bibliography (15% of final grade): For each article read, teachers wrote a one paragraph summary.
2. In class work and individual/group activities (15% of final grade). This included in-class group work on MEAs, group discussions on the readings, and individual/group presentations.
3. Homework Assignments (40% of final grade). There were six homework assignments: (a) a one page, double-spaced reflection based on what teachers learned from the engineer guest speaker. The reflection focused on questions such as “What is engineering”, “How does it relate to every lives”, and “How does engineering relate to the disciplines of mathematics and pure science”. (b) A 1-2 page evaluation of the textbooks that teachers used in their classrooms, including an outline of the content, mapping each section with the NCTM (National Council of Teachers of Mathematics) and NSES (National Science Education Standards) standards and a discussion of the usefulness of the real world context. (c) A one-day lesson plan adapting a textbook lesson to incorporate engineering ideas into the class. (d) A class presentation of one Model Extension Activity (MEA) that was created by their groups. (e) One MEA that teacher could use in his or her unit lesson plan, with a ½ page journal reflection about how this MEA could fit into the lesson plan. (f) A survey of K-12 engineering curricula and a one page summary sheet of a curriculum chosen by each teacher.
4. Unit Lesson Plan (30% of final grade): The purpose of the course was to prepare teachers to integrate engineering concepts into mathematics or/and science. Therefore, the largest percentage of their grade for the semester was the preparation of a unit outline that incorporates engineering into their subject area, two lesson plans for the unit, and a list of assessments teachers will use for this unit. The teachers were able to work with a partner on the unit

plan assignment with one who taught the same subject, science or mathematics. Teachers worked in pairs to create units; then each teacher created two lesson plans for the unit, resulting in four lessons for each unit. The assignment was broken up into five phases, which were meant to help the teachers remain organized and on track, as the course is condensed into three weeks. Partial drafts were due throughout the three weeks and the teachers were encouraged to use their time wisely and solicit feedback and ideas from their classmates. The teachers could adapt previously used lesson plans. They could also find resources on the internet or in books or use material from this class. Teachers were encouraged to include at least one MEA in the units.

### *Introduction and an Example of MEAs*

Modeling-eliciting activities (MEAs) are open-ended, client-driven problems in real world contexts (e.g. engineering tasks) that require teams to solve. In general, the problem statement of a MEA introduces students to a task. Students need to define the problem a client needs solved and create a plan of action to successfully meet the client's needs. Through the problem solving session of a MEA, students need to work as a team, purposely test, refine, and extend their plan through several documentation trails. This requires that a group of students go through multiple iterations of testing and revising their solution to ensure that their procedure or algorithm will be useful to the client (Lesh, Hoover, Hole, Kelly, & Post, 2000). The core of MEAs is the modeling perspective which differs from the core of typical problem solving activities. Students often have a lengthy interpretation phase when they struggle to create constructs that will fit the needs of clients. They discuss, paraphrase, and/or draw diagrams to try to create a mathematical model that can be described sequentially. The final product of MEAs is students' mathematical models, while the traditional problem solving activities are often focus on the creation of a physical product (Diefes-Dux, Moore, Zawojewski, Imbrie, & Follman, 2004).

Table 2 shows the Aluminum Bats problem, one MEA used in the graduate course. is in nature. This activity was developed by the Small Group Mathematical Modeling (SGMM) Project, Purdue University and has a materials science and engineering focus.



Table 2. Sample MEA – Batter, Batter, ... Swing!

**BATTER, BATTER, ...SWING!**

Osceola, IN – The Lady Panthers are ready to pounce! Coach Greg Meyers verified today that he will be forming a new summer league softball team, the Lady Panthers, for girls 12 to 13 years old.

“We have been signing up players, and we still have two positions open – third base and centerfield. So, if you know of anyone that might be interested in playing these positions or even other positions, please have them contact me,” said Meyers. “We are also beginning to make decisions about our uniforms and the pieces of equipment that we need to purchase.”

The Lady Panthers will wear uniforms of yellow and black after their team colors. Harry’s Sport Shop on Main Street is designing the uniforms, and the uniforms will be available for purchase by next Friday. Players will be responsible for purchasing their own uniforms, cleats, and mitts. Harry’s will also have available other Lady Panthers items such as baseball hats, keychains, and T-shirts for Lady Panther fans.

Since deciding on the team’s colors and the uniforms, Coach Meyer has been investigating the purchase of the necessary equipment for practice and games. He has already purchased plenty of softballs for the team and has been pricing batting helmets. Gart Brothers Sports has helmets available for \$34.99 and Outpost Sports has them available for \$32.95.

“I’ll probably purchase the helmets from Gart Brothers because they are of better quality than the helmets available at Outpost,” said Coach Meyers. “Besides, I can pick up the helmets when I also purchase the catcher’s mitt and the catcher’s mask from Gart.”

The only remaining equipment for the coach to purchase will be the softball bats. Currently, he has found three styles of aluminum bats that he likes and that cost the same amount. All three styles are available at Harry’s Sport Shop.

“Since bats are so expensive and last year the bats dented too easily, I want to purchase bats that are more resistant to denting,” commented Coach Meyers.

The first game for the Lady Panthers will occur on June 6 at home. They will be playing the Nappanee Ravens at Strawberry Field.

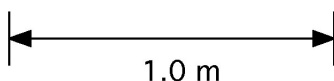
“I’m looking forward to helping the girls get ready for our first game. I’ve heard the Nappanee Ravens have some good players, so we’ll need to be ready to go!” explained Coach Meyers.

We want to wish good luck to Coach Meyers and the Lady Panthers in their game against the Ravens and in their upcoming season!! Take ‘em out with a growl, ladies!

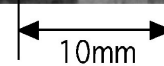
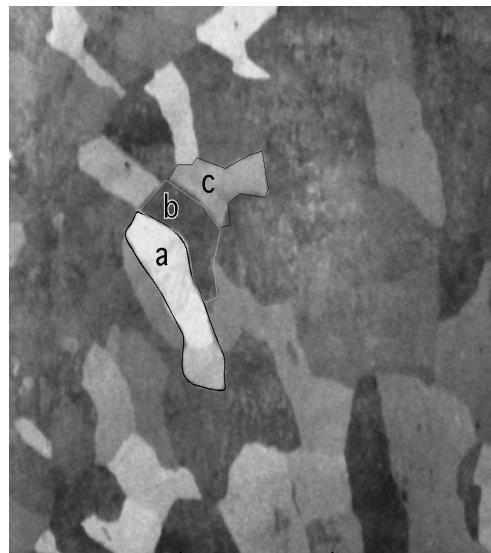
*Coach Meyers knew that Eva, who plays first base for the Lady Panthers, has an older sister that works as a materials engineer. Her name is Louisa Rodriguez, Ph.D. When he contacted Dr. Rodriguez, she explained that the size of the crystals in the aluminum is often a good indicator of the relative resistance to denting or strength of the material. She said that aluminum consisting of smaller crystals was stronger than aluminum consisting of larger crystals. Dr. Rodriguez volunteered to provide microscopic photographs of the crystal size called ‘micrographs’ because*

they were the standard way to compare the size of the crystals. Materials engineers can chemically treat polished pieces of aluminum to make the boundaries between the crystals more visible. Using a camera attached to a microscope, a picture of the boundaries between the crystals can be obtained and then the size of the crystals can be estimated.

Coach Meyers was fascinated and asked if it is ever possible to see metal crystals without a microscope. Dr. Rodriguez suggested that Coach Meyers check out the new metal poles supporting the traffic lights on a nearby corner. These steel poles are coated with a thin layer of zinc metal that helps prevent rust formation. The zinc metal forms very large crystals that can be readily seen by eye. The pictures below show the metal pole and a close-up picture of the crystals on the surface of the pole. The letters a, b and c indicate three crystals that have had a line drawn along the boundaries between the crystals. The arrow on the drawing is the scale marker for this picture.



**Figure 1: Traffic Light Pole**



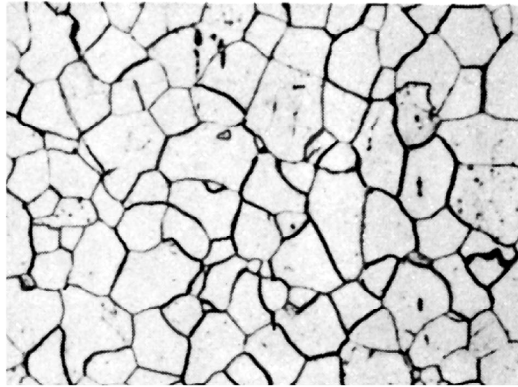
**Figure 2: Crystals**

### Readiness Questions

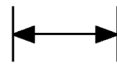
1. What positions are still open on the Lady Panther's team?
2. What equipment are the players responsible for purchasing on their own?
3. Why is Coach Meyers purchasing the batting helmets from Gart Brothers when they are cheaper at Outpost Sport?
4. How is Coach Meyers going to decide which bat to purchase?
5. How is the size of an aluminum crystal related to a bat's resistance to denting?
6. How can material engineers view crystals when they are too small to be seen by the naked eye?

7. Can some crystals be seen by the naked eye? Where?
8. Given the scale marker below the picture of the traffic light pole, how wide is the pole?

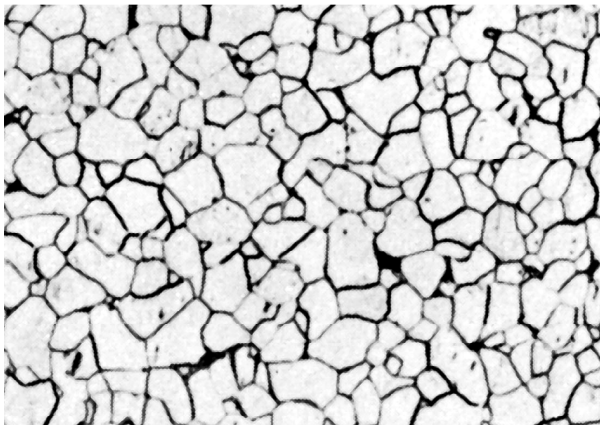
### The Choice of the Aluminum Bat



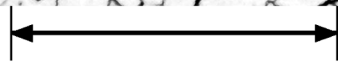
A



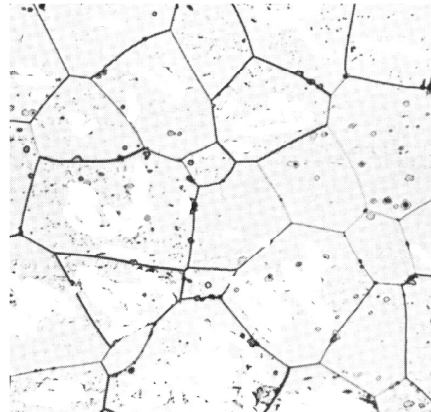
0.1mm



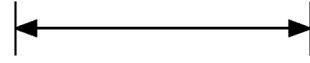
B



0.25mm



C



0.15mm

## Research Component

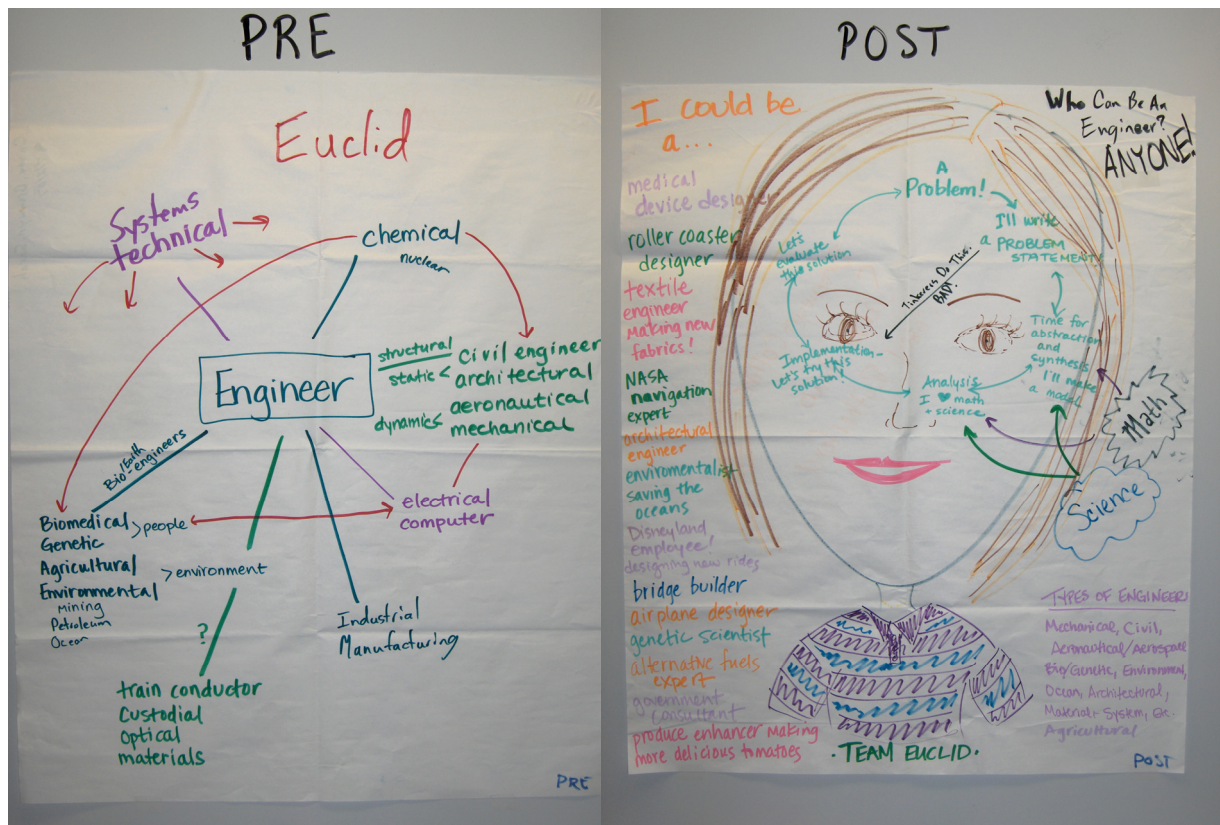
The design and development of this course was also driven by the following research questions: (1) What are the mathematics and science teachers' perceptions of the discipline of engineering while participating in a master's course on integrating engineering into their classroom? (2) How do their perceptions begin to change through this participation? (3) What are their ways of thinking regarding using engineering as a context to teach their discipline?

A qualitative and quantitative method (mix method) was chosen for data collection and analysis in order to answer the research questions. Tashakkori and Teddlie (1998) defined mix method study as "studies that are products of the pragmatist paradigm and that combine the qualitative and quantitative approaches within different phases of the research process". Miles and Huberman (1994) explained that within different phases of study there might be one or more applications. For example, a study may begin with quantitative design, followed by qualitative data collection, then convert data into quantitative data for analysis. This paper will only present a brief sample of the data, methods of analysis, and preliminary results. The 12 participants included students in this course who were in-service mathematics and science teachers at grades 5-14 (6 math teachers, 4 science teachers, and 2 taught both math and science). Data sources include: (1) Artifacts of class activities (e.g. concept maps, posters drawn by team and audio/video taped class activities/discussions); (2) Homework papers (e.g. students' reflection on "what is engineering?"); (3) Semi-structured interviews with students (e.g. students' views of integrating engineering into their classrooms); (4) Pre- and post-course surveys of students' views of the nature of mathematics, science, and engineering. Three researchers participated in coding the interviews, concept maps, and written reflections.

Preliminary findings of the changes in teachers' perceptions of engineering are presented through one student team's pre-post posters. Students in teams of three or four were asked to make a poster illustrating their understanding of engineering at the beginning and end of the course. Figure 1 shows the pre- and post-course posters from team Euclid (named by the team). This team was composed of one male and two females: Charlie had three years' high school teaching experience and taught both mathematics (geometry) and physics. Susan had eight years' experience teaching high school biology. Britta was a high school mathematics teacher with five years' teaching experience.



Figure 1. Pre/post course posters on "What is engineering" from team Euclid



The main theme from the pre-course poster (the left side of Figure 1) was that the teachers perceived engineering as compartmentalized into different professions or disciplines. The center hub labeled "Engineer" connected to seven branches or lists. The first branch connected to the lower left of the central hub is labeled "Biomedical, Genetic, Agricultural, Environmental." This branch is complex being located at the end of three interconnected arrows, one toward the central hub with the text "Bio/Earth-Engineers". To add to the complexity of this branch, the secondary labels are grouped and labeled by arrows indicating the grouping; "Biomedical and Genetic" are grouped by the label "people" and "Agricultural and Environmental" are grouped by the label "environment." A third order of complexity exists with a list "mining, Petroleum, Ocean" directly below the label "Environmental." Moving clockwise to the branch to the far upper right of the central hub labeled "Systems technical" displaying four symmetrically situated arrows pointing away from the hub. Moving clockwise is a second branch labeled "Chemical" with the text "nuclear" displayed below the secondary label. Also displayed are two arrows pointing away from the secondary label, one arrow pointing to the immediate right and another arrow pointing to the "Biomedical, Genetic..." branches. Continuing to move clockwise, the next branch displays the text "Civil engineer, architectural, aeronautical, mechanical" in a hierarchical list. The connection from the central hub to this branch is labeled "Structural." In addition an arrow connects "Civil engineer" and "architectural" is labeled "static," while an arrow connecting the last two professions in the list "aeronautical" and "mechanical" is labeled "Dynamic." There is

connection to the branch to the immediate right "electrical, computer." This branch while connected to the central hub is also displayed on one end of a two-headed arrow connecting to the "Biomedical, genetic..." branch.

It is interesting that team Euclid's post-course poster (the right side of Figure 1), displayed *the representation of a female authority*. Along the upper portion of the diagram to the left the words "I could be a..." and to the right "Who can be an engineer? ANYONE!". This indicates that teachers had a broader view on who could become engineers. In the face portion of the drawing is a representation of the engineering design process "A problem <-> I'll write a problem statement <-> Time for abstraction and synthesis, I'll make a model <-> Analysis, I 'heart' math & science <-> Implementation-let's try this solution <-> let's evaluate this solution <->". Along the left border and the word "medical device designer, roller coaster designer, textile engineer making new fabrics, NASA navigation expert, architectural engineer, environmentalist saving oceans, Disneyland employee! Designing new rides, bridge builder, airplane designer, genetic scientist, alternative fuels expert, government consultant, produce enhancer making more delicious tomatoes." Along the right border the labels "Math" and "Science" are connected to the inter diagram of the engineering design process. Additional on the bottom right border are the work "Type of Engineering," with a list "Mechanical, Civil, Aeronautical/Aerospace, Bio/Genetic, Environmental. Ocean, Architectural, Material & System, Etc., Agricultural."

Comparing the pre-course poster, which demonstrated the teachers' perceptions of engineering as limited to professions of engineering discipline, with team Euclid's post-course poster demonstrates a wider depth and range of skills that are required for engineering. The engineering design process was clearly represented at the center (head of the representation) of the poster, with math/science integrated into the process. Due to the nature of this paper, only the above preliminary results are included. More results from interview, artifacts, and pre-post course surveys will be presented in the papers for future publication.

## Discussion and Conclusion

Teacher perceptions, beliefs, and attitudes play an important role in their classroom practices and the teacher change process (Nespor, 1987; Pajares, 1992; Peck & Tucker, 1973; Richardson, 1996). One of the learning goals of this course was to help teachers understand engineering and the engineering method, describe how engineering relates their teaching subject, and learn how to integrate engineering into their subject content. The preliminary results show that prior to this course, teachers considered engineering mainly as a cluster of professions such as biochemical and environmental engineers, and teachers did not show any understanding of a relationship between mathematics, science and engineering. Through the participation of this course, teachers recognized the design process as an important component of engineering and mathematics and science are integrated into the design process of engineering. Teachers also had come to understand the nature of the design process through their own experiences with the MEAs. Such change of teacher perceptions of engineering has

impact on teachers thinking regarding using engineering as a context in teaching mathematics and science in classroom.

It is common knowledge that teachers teach in the way that they were taught when they were students (Ball, 1990). To encourage teachers to integrate STEM contexts into their subject disciplines at high schools or middle schools, the teachers' training programs need to provide integrated STEM content and cooperative learning experience for teachers. This paper presented the curriculum and preliminary research results of a graduate course developed by a science, engineering, and mathematics educator. It is the authors' intension that the communities of science education, mathematics education, and engineering education, become more collaborative and share ideas embedding engineering within/across disciplines in the higher education setting to enhance the quality of teacher training programs, teacher classroom practice, and student learning.

This paper provides details of a course curriculum that integrates engineering contexts into science and mathematics contents for in service teachers. Research has shown that collaboration among faculty of different disciplines enhances learning, and creates a higher quality of curriculum development and research (Clark et al., 1996; Eisenhart & Borko, 1991; Krajcik, Marx, Blumenfeld, Soloway, & Fishman, 2000). This course was a joint effort incorporating expertise in three areas: engineering education, science education, and mathematics education. The teaching of this course was also a collaborative effort, with the engineering educator as the main instructor and science/mathematics educators as facilitators. Such collaboration involved significant amount of time and effort. For instance, the design of the course curriculum required approximately ten hours per week from each expert over an eight-week period. Throughout this process, each expert had to read all course readings, learn concepts and knowledge in the other two disciplines relative to the course. Communication was extremely important in developing a shared understanding because each expert had different professional backgrounds. The differences in professional language and professional culture were bridged to share ideas and build joint understanding.

## References

- Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, 90 (4), 449-466.
- Clark, C., Moss, P, Goering, S., Herter, R. J., Leonard, D., Robbins, S., Russell, M., Templin, M., & Wascha, K. (1996). Collaboration as dialogue: teachers and researchers engaged in conversation and professional development. *American Educational Research Journal*. 33(1), 193-231.
- Diefes-Dux,H. A., Moore,T., Zawojewski J., Imbrie, P. K. & Follman, D. (2004). A Framework for Posing Open-Ended Engineering Problems: Model-Eliciting Activities. Proceedings of the 2004 Frontiers in Education Conference, FIE-2004, Savannah, GA.
- Douglas, E., Iversen, C. & Kalyandurg, S. (2004). Engineering in the K-12 Classroom – An analysis of current practices and guidelines for the future, ASEE Engineering K12 Center.
- Galvin, T. (2002). 2002 industry report: Training magazine's 21<sup>st</sup> annual comprehensive analysis of employer-sponsored training in the United States. *Training*. Retrieved on August 30, 2009 from [http://www.trainingmag.com/training/images/pdf/2001\\_industry\\_report.pdf](http://www.trainingmag.com/training/images/pdf/2001_industry_report.pdf)
- Johnson, D.W., Johnson, R., & Smith, K. (1998). *Active Learning: Cooperation in the College Classroom*. Edina, MN: Interaction Book Company.
- Krajcik, J., Blumenfeld, P., Marx, R., & Soloway, E. (2000). Instructional, curricular, and technological supports for inquiry in science classrooms. In J. Minstrell & E. H. v. Zee (Eds.), *Inquiring into inquiry learning and teaching in science* (pp. 283-315). Washington, D.C.: American Association for the Advancement of Science.
- Lesh, R., & Doerr, H. (Eds.) (2003), *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching*, Mahwah, NJ: Lawrence Erlbaum.
- Lesh, R., Hoover, M., Hole, B., Kelly, A., Post, T., (2000) Principles for developing thought-revealing activities for students and teachers. In A. Kelly, R. Lesh (Eds.), *Research Design in Mathematics and Science Education*. (pp. 591-646). Lawrence Erlbaum Associates, Mahwah, New Jersey.
- Miles, M., & Huberman, M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook*. (2dn ed.). Thousand Oaks, CA: Sage.
- Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies*. 19(4), 317-328.



- Norman, K. W., Kern, A., Moore, T. (2010). A call for integrating engineering through cooperative learning in the mathematics and science teacher education program. In B.Sriraman & V. Freiman (Eds), *Interdisciplinarity in the 21<sup>st</sup> century: Proceedings of The Third International Symposium on Mathematics and its Connections to the Arts and Sciences*. Moncton, Canada, Information Age Publishing, Charlotte, NC.
- Pajares, M. (1992). Teachers' beliefs and educational research: Cleaning up a messy constrect. *Review of Educational Research*, 62(3), 307-332.
- Peck, R. R., & Tucker, J. A. (1973). Research on teacher education. In R. M. Travers (Ed.), *Second handbook of research on teaching* (pp. 940-978). Chicago: Rand McNally.
- Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. *Handbook of research on teacher education*. In J. Sikula (Ed.), *Handbook of research on teacher education* (2nd ed., pp. 102-119). New York: Macmillan.
- Tashakkori, A., & Teddlie, C. (1998). *Mixed Methodology – Combining Qualitative and Quantitative Approaches*. Thousand Oaks, CA: Sage.

## **The Influence of a Multidisciplinary Scientific Research Experience on Teachers Views of Nature of Science**

*Jerine Pegg<sup>1</sup>,  
Department of Elementary Education  
University of Alberta*

*Edith Gummer<sup>2</sup>  
Center for Classroom Teaching and Learning  
Northwest Regional Educational Laboratory, Portland, Oregon*

**Abstract:** *This study examined a professional development project for K-12 science teachers that engaged participants in an authentic scientific investigation along with explicit-reflective attention to nature of science (NOS). The Views of Nature of Science (VNOS) and Views of Scientific Inquiry (VOSI) Questionnaires (Lederman, Abd-El-Khalick, Bell, & Schwartz, 2002; Schwartz, Lederman, & Thompson, 2001) were used to examine the relationship between teachers' views of NOS and specific aspects of the professional development project. Results of the study show that teachers' views of NOS were influenced by the multidisciplinary, primarily non-experimental research that they engaged in, the opportunity to observe interactions of scientists from different disciplines, and explicit classroom activities and discussions regarding NOS.*

**Keywords:** nature of science; teacher's views; teacher professional development; math and science education

### **Introduction**

Contemporary reform documents have emphasized the importance of helping students develop adequate conceptions of the nature of science (NOS) (AAAS, 1990, 1993; National Research Council [NRC], 1996). However, research has shown that teachers' understandings about NOS are frequently inconsistent with current conceptions (Abd-El-Khalick & Lederman, 2000;

---

<sup>1</sup> 551 Education South  
Edmonton, Alberta, Canada T6G 2G5  
780-492-1580  
[jerine.pegg@ualberta.ca](mailto:jerine.pegg@ualberta.ca)

<sup>2</sup> 101 SW Main St., Suite 500  
Portland, OR 97204  
503-275-9171  
[gummere@nwrel.org](mailto:gummere@nwrel.org)

Lederman, 1992). If teachers hold naïve views of NOS then they will almost certainly pass these beliefs onto students. Therefore, a necessary step in improving instruction and students' conceptions related to the nature of science involves first addressing teachers' conceptions of NOS.

Engaging teachers in research experiences has shown some success with moving teachers towards more informed views of NOS (Lord & Peard, 1995; Schwartz, Lederman, and Crawford, 2004). A few studies have begun to examine factors that enhance the impact of these research experiences, such as the use of reflective journals and explicit NOS activities (Schwartz, et al., 2004; Richmond & Kurth, 1999). This study builds on this research by specifically examining the characteristics of an authentic research experience that led teachers to re-examine their prior beliefs about NOS.

### **Theoretical Framework**

Definitions of NOS have been debated and have changed over time for philosophers of science and science educators (Abd-El-Khalick & Lederman, 2000). However, there is general agreement about the aspects of NOS that are relevant for teaching K-12 students. Eight aspects of the nature of science – tentativeness, subjectivity, creativity and imagination, observations and inferences, socio-cultural embeddedness, theories and laws, empirical basis, and multiple scientific methods (Lederman, 2004) – provided a focus for the professional development described in this study.

Models of professional development focused on enhancing inservice and preservice teachers understanding of the nature of science have taken many forms, including integration of NOS concepts into methods courses (Akerson, Abd-El-Khalick, & Lederman, 2000; Scharmann, Smith, James, & Jensen, 2005; Tairab, 2002; Ogunniyi, 1983), courses focused on the history and philosophy of science (Abd-El-Khalick, 2005; Akindehin, 1988), and professional development projects involving research experiences with scientists (Schwartz, Lederman, & Crawford, 2004).

Many teachers at the elementary and secondary level have never been directly engaged in scientific research. Their view of science has come primarily from K-12 and university level coursework that often focuses on the concepts and skills of science rather than the nature of the scientific process. Engaging teachers in research with scientists places them within the community of practice of science (Lave & Wegner, 1991). When teachers experience authentic inquiry with scientists they engage in peripheral participation that allows the teachers to speak with the scientists about their activities, identities, artifacts, knowledge and practice. This engages participants in learning "... who is involved; what they do; what everyday life is like; how masters talk, walk, & generally conduct their lives ..." (Lave & Wegner, 1991, p. 95).

Although, few studies have examined the influence of authentic science on students' and teachers' understanding of NOS (Schwartz & Crawford, 2004), benefits of research experiences have been documented for students, preservice teachers, and inservice teachers. Undergraduate students engaged in research experiences gained a better understanding of scientific methods (Kardash, 2000). Research experiences have also been found to increase teachers' content knowledge (Buck, 2003, Dresner & Worley, 2006; Raphael, Tobias, & Greenberg, 1999) and confidence with teaching inquiry science (Westerlund, Garcia, Koke, Taylor, & Mason, 2002).

Studies that have examined the impact of research experiences on teachers' and students' views of NOS have shown mixed results. Some studies have shown positive impacts of research experiences on students' and teachers' views of NOS (Lord & Peard, 1995; Richmond & Kurth,

1999; Schwartz, et al., 2004), while others have shown no effect (Buck, 2003; Bell, Blair, Crawford, & Lederman, 2003). Although participants were engaged in research experiences in all of these studies, the studies which found changes in participants' views of NOS also incorporated strategies for making aspects of the nature of science explicit and engaging participants in reflection on their experiences (Lord & Peard, 1995; Richmond & Kurth, 1999; Schwartz, et al., 2004).

Schwartz and Crawford (2004) suggest that research experiences can be used to teach NOS provided that critical elements are integrated with the experience; (1) explicitly treat NOS as content, (2) facilitate reflection, (3) understand that one does not "do NOS". The research experiences provide an authentic context for reflection that is a necessary, but not a sufficient condition for challenging teachers' views of NOS.

Previous studies suggest that research experiences can improve participants' views of NOS when explicit-reflective attention is paid to NOS (Schwartz & Crawford, 2004). An implicit assumption of this argument is that the research experience provides the necessary context for participants to reflect upon their experiences. However, research experiences can vary widely. Furthermore, aspects of NOS (Lederman, 2004) are based upon a characterization of science as a whole and may not be observable in all research experiences. In order to better understand how research experiences may inform teachers' views of NOS, this study examined the relationship between the nature of the research experiences in a professional development project for teachers and teachers' views of NOS.

#### Description of the Professional Development Program

The Mammoth Park Project was a professional development project for K-12 science teachers that engaged teachers in an authentic scientific investigation along with explicit-reflective attention to NOS. The project involved collaboration between scientists, science educators, and teachers focused on developing teachers' knowledge of NOS, scientific inquiry (SI), and related science content, as well as supporting teachers to implement interdisciplinary inquiry-based instruction. The project consisted of two-days of introductory workshops in the spring, a two-week summer field session, and three Saturday workshops during the following school year. This study was conducted in the fourth year of the project and focuses on the research experiences and associated activities that occurred during the summer workshop.

The professional development model examined in this study varies from many other research experiences for teachers and students. The participants in this project were engaged in multidisciplinary research, they were engaged in all aspects of the investigation from defining questions to interpreting results, and explicit-reflective attention to NOS was integrated with the research experiences.

#### *Multidisciplinary, Primarily Non-experimental Research*

In the Mammoth Park Project scientists from multiple disciplines worked together to investigate the paleoecological history of the Willamette Valley in Oregon. The scientists involved in the project included two paleoecologists, two archaeologists, a geologist, an entomologist, and a physicist. The primary goal of the project was to locate peat deposits at the study sites and to examine the peat for evidence of the climate conditions and presence of organisms present at the time being studied. The two paleoecologists involved in the project specialized in diatom and charcoal analysis. The entomologist assisted with identification of insects in the peat. The geologist specialized in soil identification and analysis of stratigraphy. The archaeologists were present to assist with survey techniques and identification of artifacts if

any were found. The nature of the research that most of the scientists were conducting was primarily observational and non-experimental. However, the physicist's research was experimental and involved testing hypotheses about the relationship between magnetometry readings and the location of peat deposits. During the project, the teachers interacted with the entire research team rather than being placed with individual scientists. This context provided a multidisciplinary perspective that included experimental and non-experimental research.

#### *Teachers as Co-investigators*

During the Mammoth Park Project, scientists and science educators worked together to engage teachers in the investigations that the scientists were conducting in a way that immersed the teachers in the experience without overwhelming them with content or the procedural aspects of the investigation. Teachers were considered co-investigators along with the scientists and were engaged in all aspects of the investigation including defining questions, designing procedures, collecting data, and interpreting results.

The scientists were engaged in actual research related to their individual research programs. However, for the Mammoth Park Project, the scientists were brought together primarily for the purposes of the professional development project for the inservice teachers. Therefore, the scientists' decisions about the directions of the research could not be purely based on their own professional decisions, but rather included collective engagement with the other scientists, the science educators, and the teachers. In many ways this makes the research more "authentic" for the teachers as they were allowed to experience science as co-investigators with the scientists from the very beginning.

#### *Explicit-Reflective Attention to NOS*

The Mammoth Park Project was designed by the scientists and science educators to weave activities about NOS and SI throughout the project. In the spring, teachers attended an introductory classroom workshop that introduced them to the focus of the research and engaged them in activities related to aspects of NOS, including observations, inferences, and models. Thus providing the teacher-participants with a background and glimpse of what would be present during the research they would be involved with in the summer.

During the two-week summer institute, additional activities were chosen to address aspects of NOS and explicitly connect to the paleoecological research. The NOS activities were connected to the scientific field and laboratory activities in a deliberate manner to provide the teachers with a concrete connection between the scientific investigation they experienced and the NOS exercises that they might use in the classroom. Explicit discussions of SI focused on the essential features of classroom inquiry identified in the book "Inquiry and the National Science Education Standards" (National Research Council, 2000). This was addressed in a classroom session during the summer workshop and in informal discussions as teachers and scientists engaged in the paleoecological investigation. For example, the science educators continually encouraged scientists and teachers to examine the nature of the evidence that supported claims they were making. Since the scientists and science educators were integrally involved in all of the project activities they were able to take advantage of "teachable moments" that occurred during teachers engagement in the actual research and help teachers make connections between issues that arose during the research and aspects of NOS.

## Methodology

An interpretive case study approach (Merriam, 1998) was utilized to examine the relationship between teachers' views of NOS and their experiences in an authentic research experience. The primary researcher was a participant observer in the professional development. This allowed the researcher to place the teachers' conceptions within the context of the professional development that they were engaged in.

### Participants

Twenty-five inservice teachers participated in the Mammoth Park Project during the fourth year of the project. Fourteen of these teachers completed the pre-, mid- and post-questionnaire and were included in this study. There were ten females and four males. The teachers taught grades ranging from 4th through 12th and the length of their teaching experience ranged from 1 to 24 years (Table 1).

Table 1

### Background Information for the 14 Participants in this Study

Name	Grade Taught	Subject Area Taught	Years of Teaching Experience
Sarah	4th	All subjects	2
Deborah	4th	All subjects	9
Matt	5th	All subjects	4
Todd	5th	All subjects	13
Kelly	5th	All subjects	17
Heather	5th	All subjects	24
Brenda	7th	Life Science	1
Nancy	7th	Science and Math	23
Allison	8th	Integrated Science	3
Peter	8th	Earth and Physical Science	6
Paul	8th	Integrated Science	10
Lisa	9th	Physical Science	3
Rebecca	10th	Biology	1
Melinda	9th – 12th	Astronomy, Geology, Environmental Science, Biology	20

*Note.* Names are pseudonyms.

### Data Sources

The VNOS-C' and VOSI Questionnaires (Lederman, Abd-El-Khalick, Bell, & Schwartz, 2002; Schwartz, Lederman, & Thompson, 2001) were used to examine teachers views of NOS prior to, during, and following the Mammoth Park Project. The VNOS-C' and the VOSI are open-ended questionnaires. The open-ended nature of the questionnaires provided teachers with the opportunity to explain their views and to connect their explanations to personal experiences, including those from the professional development.

For this study, analysis of the VOSI focused on the aspect of multiple scientific methods and provided supporting evidence for the other seven aspects of NOS. The VOSI questionnaire has questions that address additional aspects of SI, such as the distinction between data and evidence. However, these additional aspects of SI are not addressed in this study.

Teachers completed these questionnaires prior to (pre-survey) and following (post-survey) the summer workshop. In addition, one question from the VNOS-C' and one question from the VOSI were given to the teachers at the end of each day during the summer workshop to examine how the activities of each day were influencing their views. In the following discussion, this is referred to as the mid-survey. This study focuses on data from the mid and post-survey.

Because the teachers had the opportunity to engage with the instrument probes and questions on multiple occasions, before, during, and after the project, the instruments became instructional tools as well as evaluation tools. Previous studies have noted that the act of completing the VNOS survey may result in deeper reflection and clarification of beliefs (Bell et al, 2003; Lederman & O'Malley, 1990). This gave the teachers repeated opportunities to consider how their understanding was influenced by their involvement in the paleoecological research.

### *Data Analysis*

Teachers' responses on the VNOS-C' and VOSI from the mid- and post-surveys were analyzed to identify relationships between the teachers' characterizations of NOS and SI and specific aspects of the professional development. The surveys were examined question by question for each teacher. Statements from the mid- and post-survey that specifically referred to examples from the teachers' experiences during the summer field session were identified. This resulted in 60 statements that explicitly referenced the project. Analysis of these statements consisted of two levels of coding. First, the aspect of NOS or SI to which they referred was identified. Second, an inductive analysis was used to identify the aspects of the project that were referenced. Codes related to the aspects of the project emerged from the data. For example, the following response to the question, "What types of activities do scientists do to learn about the natural world?" was coded as "Scientific Methods" and "Non-experimental Inquiry":

Make observations. Study test samples and collections. Make comparisons to known samples or comparative studies – Ex. We talked a lot about seeing if other soils in valley match the river, etc. Use of maps: elevation, contours, locations, topography, lab work/tests on samples. -> Ask questions and provide evidence to explain. (Peter, VOSI#1, Mid-survey)

The coded statements were then examined and three themes were identified that related to the relationship between the aspects of NOS and the teachers references to the project.

### **Findings**

Analysis of teachers' references to the project resulted in the identification of three themes related to relationships between aspects of NOS and specific components of the professional development model.

#### *Non-experimental, Multidisciplinary Inquiry – Multiple Scientific Methods*

The primarily non-experimental nature of the investigation provided teachers with examples of scientific methods that were very different from the experimental methods that many of them were familiar with. Furthermore, the multidisciplinary nature of the investigation

provided teachers with examples of various scientific methods that allowed them to compare and contrast the differing methods. As one teacher stated, “The work on the Mammoth Park Project was an example of different scientists with different points of view and different methods, investigating a paleontology dig” (Todd, VOSI #5, Post-survey). Another teacher explicitly pointed out how the research that he experienced during the Mammoth Park Project did not follow the traditional scientific method often illustrated in textbooks. He stated that “Mammoth Park has not been the public educational model of science: state the problem, hypothesis, experiment, collect data, conclusion, etc. We have been a part of multiple problems being identified by various scientists” (Matt, VOSI#5, Mid-survey).

Teachers referred to examples from the Mammoth Park Project to describe how science includes observational methods and experimental methods. One teacher described how the observational nature of the Mammoth Park Project differed from his dominant view of science:

So far we haven’t tested any hypothesis that I know of – and this has been my dominant view of science: form a question, make a hypothesis, and perform experiments/tests that will attempt to answer the question. ... The digging we’re planning on performing on Thursday forward seems to be more data collection in an attempt to make sense of the observations. I’ve witnessed very few hypotheses being entertained so I’m curious to see if this happens (Matt, VNOS #1, Mid-survey).

This response was collected from the participant following the first day of the project. Later in the week, teachers had the opportunity to interact with the physicist on the project who had developed specific hypotheses that were guiding his investigations. This is discussed in more detail below.

Another teacher, who described the things that scientists do as basically following the scientific method in her pre-survey, used specific examples from Mammoth Park in her mid-survey to provide a much richer description of the activities of scientists. She stated that:

They physically go to the site in question. They will take samples of what they are studying: core samples, plant samples, insect samples, soil samples. They will consult past and present studies of the site. They will research any collected data of the site (Melinda, VOSI#1, Mid-Survey).

The multidisciplinary nature of the investigation also provided an opportunity for teachers to see scientists using various methods in their approaches to the research questions. The paleoecologists, geologists, and archaeologists used primarily observational methods that were not always guided by a priori hypotheses. On the other hand, the physicist had a specific hypothesis about the relationship between magnetometry readings and the location of peat deposits. One teacher used this contrast between the types of research to describe how science does not follow one scientific method:

At Mammoth Park Project this past summer, I saw this actually play out. As we began our process, [the geophysicist], announced his hypothesis for his portion of the project. The data that we would eventually collect would support or not support his hypothesis. While talking to the other scientists on the project, they were waiting to see what the data showed prior to setting a hypothesis. Both are scientific because the data was collected systematically and both will still lead to conclusions/explanations based upon the evidence/data gathered in the project (Peter, VOSI #5, Post-survey).



*Interactions of Scientists from Different Disciplines – Subjective and Socio-cultural NOS*

The multidisciplinary nature of the Mammoth Park Project allowed teachers to see how scientists from different disciplines approached questions, designed investigations, and interpreted information. Teachers' references to the Project cited multiple examples of how the interactions of the different scientists were representative of the subjective NOS and the socio-cultural embedded-ness of science.

When asked whether or not scientists could reach different conclusions when asking the same question or interpreting the same data (VOSI #6, VOSI #7, and VNOS #7) six teachers described specific examples from the Mammoth Park Project that were representative of how the different backgrounds and motivations of scientists could influence their interpretations of data. For example, one teacher stated "The same way that [the Archaeologist's] explanation of the gravel deposits at the river level of [location] differed from [the Geologist's] explanation. The data/evidence was the same but their experiences differed enough to influence their view of that evidence" (Matt, VNOS #7, Post-survey).

The interactions of the scientists gave the teachers the opportunity to see how different backgrounds and perspectives can lead to different conclusions. The interactions also gave the teachers the opportunity to see how communication among scientists allowed them to discuss their differing conclusions and the evidence that they used to develop their conclusions. In some cases, they were also able to observe how discussion among the scientists allowed them to reach consensus about how the evidence fit the conclusions. One teacher described an incident where scientists and teachers discovered an unknown white layer in a trench that was dug on one of the sites. Lisa described how she observed two of the scientists come to different conclusions after observing the white layer and how they eventually reached a consensus after discussing their reasoning, "[Archaeologist #2] explained why he concluded it was an ash deposit. [Archaeologist #1] reconsidered and agreed that it was most likely an ash deposit. In other situations, scientists might still draw different conclusions" (Lisa, VOSI #6, Post-survey).

When discussing socio-cultural influences on science, teachers focused primarily on issues of social influences including how scientists choose what to study, comparisons between different scientists, or comparisons between different disciplines of science. When asked about how scientists choose what to study, Matt used specific examples from discussions with the scientists and science educators on the project to describe a variety of factors:

Factors that I've heard from the scientists on the project: 1. Funding sources – [physicist's] use of the magnetometer influenced by simply having the instrument available via grant. 2. Opportunities available because parents exposed the scientist to specific types of science ([science educator] and her grandfather's influence). 3. Demands of the organization. If a department says do this and the scientist wants to remain in the organization, then they do the study. 4. Personal interest of the scientists ([archaeologist] chose not to study pollen because it wasn't of personal interest to him) (Matt, VOSI#2, Mid-survey).

Teachers also focused on how the different perspectives of individual scientists can reflect social values. Kelly stated that, "Science reflects social and cultural values. Different individuals reflect different social values and interests. Example: the scientists working at Mammoth Park Project..." (VNOS#10, Mid-survey). Deborah specifically highlighted the

difference between different scientific disciplines. On her mid-survey she stated "... the culture of physicists is different from the culture of archaeologists" (VNOS#10).

Teachers' references to cultural influences on science focused on discussions they had with scientists and science educators during the project, rather than direct observations of how cultural differences could influence science. For example, on her mid-survey, Allison stated that her view on the influence of social and cultural factors was changing and that this was primarily due to explicit discussions with one of the science educators:

OK, so I changed my answer on this one. I'm starting to see that science reflects social and cultural values, but that's in a large part due to [science educators'] explicitly saying that today. I did see some of the social aspect in the issues that came up ... in regards to scientists being on different pages with their thinking. But, I still don't really have a picture of the cultural aspect since we didn't really do anything with that. (VNOS#10)

One of the teachers explicitly described a story that the science educator told about an event that occurred in Russia. In response to VNOS #10, which asks whether or not science reflects social and cultural values, she stated that:

Scientists' social, political, and religious selves are a part of their science. An example is the story [the science educator] told of the scientists who were sent to Siberia in Russia because their crop outcome was not desirable. That fear created would have a huge impact on other scientists – and their methods and outcomes (Sarah, VNOS #10, Mid-survey).

A number of teachers also explicitly referenced differences between Western science and Native American science when describing the influence of socio-cultural factors. One of the scientists on the Project had done extensive work with local communities in Alaska and shared multiple examples of differences between Western ways of knowing and Native American ways of knowing.

The multidisciplinary nature of the project provided teachers with experiences in which they could observe how the different individual and disciplinary backgrounds of the scientists influenced their approaches to problems, their methodologies, and their interpretations. This was one more aspect of their exposure to the complexities of NOS.

### *Explicit NOS Activities – Theories and Laws & Tentative NOS*

Teachers' responses specifically referred to NOS activities that were integrated throughout the two-week summer workshop. References to the explicit NOS activities related primarily to descriptions of the differences between theories and laws.

During one of the classroom sessions, teachers were engaged in a discussion of common misconceptions about the nature of theories and laws. Specifically, teachers were asked to reexamine the commonly held view of a hierarchical nature of theories and laws. This view holds that as more and more evidence is gathered, hypotheses develop into theories and theories develop into laws. Alternatively, teachers were exposed to the idea that theories and laws are actually two different, but equally valid types of knowledge. Specifically, it was discussed that laws describe the relationships among observable phenomena, while theories explain the observable phenomena. Furthermore, theories can support laws by providing possible explanations for the observed phenomena that the laws describe, but theories do not become laws.

A number of the teachers' responses to the VNOS question about theories and laws explicitly referenced the definition of theories and laws that was discussed in the classroom sessions. For example, on his post-survey Matt stated that "According to [a science educator] there is a difference -> Laws state, identify and/or describe the relationships among observable phenomena. Theories are inferences that explain the observable phenomena..." (VNOS #5, Post-survey). Another teacher clearly described how his view had changed due to the new information that had been shared with them:

I used to believe that a theory developed into a law, but now, with new knowledge, I (and many others) am led to believe that there is an inherent difference. I used to say if a theory was beyond any known refute, then it could develop into a law. Now, as I understand it, they are based on two different corresponding details (Paul, VNOS #5, Post-survey).

However, for some teachers, their understanding of the nature of theories and laws showed inconsistencies. A number of teachers expressed an understanding that theories and laws were different types of knowledge while still holding a naïve conception that laws are proven or that theories become laws. These teachers seemed to be restating what they had been told about the nature of theories and laws without fully conceptualizing the distinction. For example, Peter stated on his post-survey, "A theory is not yet accepted as truth, but is still being challenged. Scientific law is accepted as truth. It has been time tested over and over" (VNOS#5). He went on to provide examples of how the theory of plate tectonics is still being challenged and modified, but the laws of motion are accepted as a fact. Then at the end of his response, he identified the following as the "Mammoth Park Project Description: ...Law – describes a relationship among observable phenomena. Theory – inferred explanation for observable phenomena". Peter had apparently learned this distinction through his participation in the project, but he had not fully incorporated it into his understanding of theories and laws, nor had it influenced his more naïve view that laws have been proven true and theories are still being challenged.

The teachers appeared to be able to recall the definitions that were discussed during the classroom session, but had trouble integrating these definitions with their former conceptions of a hierarchical nature of theories and laws. For example, on her post-survey, Deborah referred to the definitions that she learned, but then appeared to revert to a hierarchical description of theories and laws when she was unable to refer directly to her journal notes:

Yes, there is a difference between a scientific theory and a scientific law. If I had my handy dandy journal I could give you an example of each. However, since I've turned it in, I can't remember exactly. ...A law can be reproduced and will always be reproduced and will always have the same outcome – it can be proven with the evidence. A theory may not be reproducible. It may not have the same outcome each time (VNOS#5, Post-survey).

These results suggest that for a number of teachers, explicit discussions of aspects of NOS during the Mammoth Park Project made them more aware of the distinction between theories and laws. However, these explicit discussions did not necessarily move teachers toward fully informed views of this aspect of NOS.

## **Discussion**

Research experiences can provide teachers an opportunity to learn more about the practices of science and the nature of the scientific process. Through these experiences scientists often mentor teachers about scientific practices (i.e. methods of data collection and

interpretation). However, the aspects of NOS are often more implicit and may not be observed or recognized unless the appropriate experiences and reflective context are provided as a component of the teachers' experiences. This study found that the multidisciplinary and primarily non-experimental nature of the research experiences and the explicit-reflective attention to NOS provided a context that challenged teachers' views of NOS.

During the project, the teachers worked with scientists from various disciplines who utilized different methods in their approaches to investigating their particular research questions. The observational nature of the investigations that occurred during the Mammoth Park Project challenged many of the teachers' views of what counts as "science". Involvement in the research gave them specific examples of scientific investigations that differed from their view of the "scientific method" and the single variable experiments with which they were familiar.

Teachers' views of the subjective and socio-cultural NOS were also enhanced by the multidisciplinary nature of the Mammoth Park Project which provided them with the opportunity to observe interactions among scientists from different disciplines. Teachers observed scientists reach different conclusions when examining the same evidence due to differences in their previous knowledge and perspectives. Their interactions with scientists in this project also allowed teachers to directly observe the role that communication among scientists plays in formulating and negotiating conclusions from data. A similar study which engaged teachers in research experiences with scientists conducted by Schwartz, Westerlund, Koke, Garcia, & Taylor, (2003) found that the teachers changed little in regards to their views about subjectivity and socio-cultural aspects of NOS. Schwartz et al. (2003) found that teachers moved from seeing science as culture-free to acknowledging the role of funding on *what* science is done. However, teachers did not recognize the role of culture on *how* science is done. The multidisciplinary nature of this project provided a context that gave teachers direct experience with the role of different scientific cultures (i.e. scientific disciplines) on how science is done and engaged them in informal discussions about how research can differ across cultures.

Changes in teachers' views of theories and laws appeared to be influenced primarily by specific discussions during the workshop rather than experiences arising from involvement in the paleoecological investigation. However, these discussions did not appear sufficient to create conceptual change related to this concept among many of the teachers. Most of the teachers attempted to add the view that theories and laws are different types of knowledge to their prior conception that theories become laws. Although the teachers were told that theories do not become laws, it appears that merely presenting this information was not enough to challenge the strongly held belief in a hierarchical relationship between theories and laws. Dagher, Brickhouse, Shipman, & Letts (2004) showed that even extended instruction at the undergraduate level related to scientific theories often fails to change students' hierarchical views of the relationship between theories and laws. The information presented in this project apparently failed to create the necessary dissatisfaction with the prior belief that would be needed in order to encourage replacement with a more informed view. Instead, teachers attempted to assimilate the new information in conjunction with their prior conceptions. Schwartz et al. (2003) also found that teachers in their project moved towards an understanding of theories and laws as different while still holding a hierarchical view of their relationship.

In this project, the teachers were engaged as co-investigators along with the scientists in an authentic science experience. Teachers were placed in a position where they were directly involved in the research process and had the opportunity to interact with and observe the other scientists as the research progressed. The scientists were forced to make their process more

explicit to the teachers in order to truly engage them in the research experience. The scientists did more than just train the teachers to collect data, they openly discussed each step in the research in order to engage and involve the teachers in the decision making process. This made aspects of the processes that the scientists used, including the multiple scientific methods and subjective nature of the scientists interpretations of data, apparent to the teachers.

The explicit-reflective attention to NOS was embedded throughout the project in formal and informal ways. The classroom activities that explicitly addressed aspects of NOS were most directly apparent to the teachers. A number of informal conversations related to NOS also occurred between teachers, scientists, and science educators. The teachers' references to the socio-cultural aspect of NOS often referenced these informal discussions. In most projects involving research experiences the scientists are not directly involved in aspects of the professional development that occurs outside of the scientists laboratory. In this project, scientists were involved in all aspects of the project including the classroom sessions focused on NOS and other pedagogical issues. This allowed the scientists to integrate discussions of NOS and SI into the actual research experience when opportunities arose.

### Conclusion

Professional development experiences that aim to enhance teachers' views of NOS need to consider both the specific context of the research experiences that teachers engage with and the nature of opportunities for explicit-reflective attention to NOS. Research experiences that engage teachers in non-experimental and multidisciplinary inquiry can provide a critical contrast to traditional representations of "the scientific method". Engaging in scientific inquiry with scientists from different disciplines can also provide opportunities for teachers to directly observe aspects of the subjective and socio-cultural nature of science. Furthermore, embedding explicit-reflective attention to NOS throughout such a project will support teachers in critically assessing their understanding of the nature of science and its relationship to the scientific research they are experiencing.

### References

- Abd-El-Khalick, F. (2005). Developing deeper understandings of nature of science: The impact of a philosophy of science course on preservice science teachers' views and instructional planning. *International Journal of Science Education*, 27(1), 15-42.
- Abd-El-Khalick, F. & Lederman, N. G. (2000). Improving science teachers' conceptions of nature of science: A critical review of the literature. *International Journal of Science Education*, 22(7), 665-701
- Akerson, V. L., Abd-El-Khalick, F., Lederman, N. G. (2000). Influence of a reflective explicit activity-based approach on elementary teachers' conceptions of nature of science. *Journal of Research in Science Teaching*, 37, 295-317.
- Akindehin, F. (1988). Effect of an instructional package on preservice science teachers' understanding of the nature of science and acquisition of science-related attitudes. *Science Education*, 72, 73-82.
- American Association for the Advancement of Science. (1990). *Science for all Americans*. New York: Oxford University Press.
- American Association for the Advancement of Science. (1993). *Benchmarks for science literacy: A Project 2061 report*. New York: Oxford University Press.

- Bell, R. L., Blair, L., Crawford, B., & Lederman, N. (2003). "Just do it." The impact of a science apprenticeship program on high school students' understandings of the nature of science and scientific inquiry. *Journal of Research in Science Teaching*, 40(5), 487-509.
- Buck, P. (2003). Authentic research experiences for Nevada high school teachers and students. *Journal of Geoscience Education*, 51(1), 48-53.
- Dagher, Z. R., Brickhouse, N. W., Shipman, H., & Letts, W. J. (2004). How some college students represent their understandings of the nature of scientific theories. *International Journal of Science Education*, 26(6), 735-755.
- Dresner, M. & Worley, E. (2006). Teacher research experiences, partnerships with scientists, and teacher networks sustaining factors from professional development. *Journal of Science Teacher Education*, 17(1), 1-14.
- Kardash, C. M. (2000). Evaluation of an undergraduate research experience: Perceptions of undergraduate interns and their faculty mentors. *Journal of Educational Psychology*, 92, 191-201.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, UK: Cambridge University Press.
- Lederman, N. G. (1992). Students' and teachers' conceptions of the nature of science: a review of the research. *Journal of Research in Science Teaching*, 29, 331-359.
- Lederman, N. G. (2004). Syntax of nature of science within inquiry and science instruction. In L.B. Flick & N.G. Lederman (Eds.), *Scientific Inquiry and Nature of Science* (pp. 301-317). Dordrecht, The Netherlands: Kluwer.
- Lederman, N. G., Abd-El-Khalick, F., Bell, R. L., & Schwartz, R. S. (2002). Views of nature of science questionnaire (VNOS): Toward valid and meaningful assessment of learners' conceptions of nature of science. *Journal of Research in Science Teaching*, 39(6), 497-521.
- Lederman, N. G., & O'Malley, M. (1990). Students' perceptions of tentativeness in science: development, use, and sources of change. *Science Education*, 74(2), 225- 239.
- Lord, T. R. & Peard, T. L. (1995). Scientist-teacher summer workshops can enhance constructivist views about science and science instruction. *Education*, 115(3), 445-447.
- Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.
- National Research Council. (1996). *National Science Education Standards*. Washington, DC: National Academy Press.
- National Research Council. (2000). *Inquiry and the National Science Education Standards*. Washington, DC: National Academy Press.
- Ogunniyi, M. B. (1983). Relative Effects of a History/Philosophy of Science Course on Student Teachers' Performance on Two Models of Science. *Research in Science and Technological Education*, 1(2), 193-199.
- Raphael, J., Tobias, S., Greenberg, R. (1999). Research experience as a component of science and mathematics teacher preparation. *Journal of Science Teacher Education*, 10(2), 147-158.
- Richmond, G., & Kurth, L. A. (1999). Moving from outside to inside: High school students' use of apprenticeship as vehicles for entering the culture and practice of science. *Journal of Research in Science Teaching*, 36, 677-697.

- Scharmann, L. C., Smith, M. U., James, M. C., & Jensen, M. (2005). Explicit reflective nature of science instruction: Evolution, intelligent design, and umbrellaology. *Journal of Science Teacher Education*, 16, 27-41.
- Schwartz, R. S., & Crawford, B. A. (2004). Authentic scientific inquiry as context for teaching nature of science. In L.B. Flick and N.G. Lederman (Eds.), *Scientific Inquiry and Nature of Science* (pp. 318-355). Dordrecht, The Netherlands: Kluwer.
- Schwartz, R. S., Lederman, N. G., & Crawford, B. A. (2004). Developing views of nature of science in an authentic context: An explicit approach to bridging the gap between nature of science and scientific inquiry. *Science Education*, 88(4), 610-645.
- Schwartz, R. S., Lederman, N. G., & Thompson, T. (2001, March). *Grade nine students' views of nature of science and scientific inquiry: The effects of an inquiry-enthusiast's approach to teaching science as inquiry*. Paper presented at the meeting of the National Association of Research in Science Teaching, St. Louis, MO.
- Schwartz, R. S., Westerlund, J., Koke, J., Garcia, D., & Taylor, T. (2003, March). Explicit/reflective NOS instruction and authentic science research: Effects on teachers' NOS views. Paper presented at the annual meeting of the National Association for Research in Science Teaching, Philadelphia, PA.
- Tairab, H. H. (2002). Pre-service teachers' views of the nature of science and technology before and after a science teaching methods course. *Research in Education*, 65, 81-87.
- Westerlund, J. F., Garcia, D. M., Koke, J. R., Taylor, T. A., & Mason, D. S. (2002). Summer scientific research for teachers: The experience and its effect. *Journal of Science Teacher Education*, 13(1), 63-83.

## Epilogue

*"We publish while others perish"*  
-Howard Zinn (1922- 2010)

*Bharath Sriraman*  
*The University of Montana*

The table of contents of this double issue included the above quote from the historian Howard Zinn which might seem puzzling to the reader. Why was this quote included and what is it supposed to mean? In the opening editorial, I mused over the whole enterprise of scholarly publishing and what it amounts to in the grand scheme of things. Zinn's quote reminds us that academia is a cloistered unit and many of the things we place importance on in the academic culture of *publish or perish* seem insignificant when viewed through the lens of *real* problems that occur outside the academic cloister. It can also mean that *we* are sitting in a position of privilege in our ivory tower offices while *others* are not.

This suggests that while we engage in scholarship we find intrinsically meaningful, often to seek the validation of peers within academic sub cultures, the prestige of publications, and to climb the meritocratic laced rungs within university systems, we do have the burden of impacting issues "outside" the culture of scholarship, which are beyond the realm of specialized discourses cloaked in domain specific vocabulary.

In the editorial I alluded to the mantra for measurable "change" that dominates discourses within mathematics education, whereas a more meaningful and non measurable change could simply be creating a sense of agency, or changing dominant discourses, or creating subtle shifts in people's perspectives.

The world of words and ideas wields more power and influence than we imagine. They can influence the discourses that occurs in the future, shape the intellectual character and fortitude of the present and coming generations. So the very least we can do is plant the seeds of "change", embrace *intellectualism* for what it really means, and be bold enough to engage in scholarship that challenges the status quo in institutional, political and economic mechanisms that characterize the academic cloister, the machinery of grant funding, of publishing and most importantly the world outside.

The most rewarding aspect of running this journal is the correspondence I received from readers over the last 8 years. Many come from teachers who are inspired to teach a mathematical idea not prescribed by the curriculum or the textbook, or implement/test a research finding reported in the journal. Then there are e-mails from undergraduate and graduate students who begin an investigation stemming from ideas published in the journal. Last but not least, the letters that motivate me to keep this journal running are those that say:



“I learned something new”

“Now I know where that mathematical idea comes from”

“It’s nice to be able to read something useful in a journal free of charge”

### **Some Closing thoughts**

I do not necessarily agree with educational research that funds and justifies itself by invoking the *needs* of one country such as the U.S to maintain its global leadership over the others because such arguments are based on the false premise of technological supremacy that fuels the educational-industrial-military economic model. Nevertheless the journal does not squelch such voices in order to push forth any particular ideology (neo capitalist, neo Marxist, neo socialist, neo progressive or otherwise). Intellectual discourse does not occur when everyone is in agreement, or afraid to be the voice of dissent, or when scholars adopt neutral stances, but occurs in a climate which tolerates differing viewpoints. We welcome readers to challenge assumptions and critique arguments put forth by authors that publish in this journal and continue to initiate change in the mindsets and orthodoxies that characterize academia. This, I believe is both our privilege and prerogative.